Adjoints

If V is a real inner product space, and $T : V \to V$ is a linear transformation, then a linear trans $S : V \to V$ is called the adjoint to T if we have

$$\forall\, v, w \in V, \qquad \langle T(v), w \rangle = \langle v, S(w) \rangle.$$

Example: Suppose V is f.d. with orthonormal basis $B = \{v_1, \dots, v_m\}$. Consider matrix

$$(a_{ij}) = M(T)_{B \to B}$$

So $\quad T(v_j) = \sum_{k=1}^{m} a_{kj} v_k \quad$, so $\quad a_{kj} = \langle v_k, T(v_j) \rangle$

Consider the transpose matrix $(M(T)_{B\to B})^T$.

There is a unique linear trans $S : V \to V$ so that $M(S)_{B\to B} = (M(T)_{B\to B})^T$. This means

$$\begin{aligned}
\langle v_c, S(v_d) \rangle &= (c,d) \text{ entry of } M(S)_{B\to B} \\
&= (d,c) \text{ entry of } M(T)_{B\to B} \\
&= \langle v_d, T(v_c) \rangle \\
&= \langle T(v_c), v_d \rangle
\end{aligned}$$

It follows $\langle v, S(w) \rangle = \langle T(v), w \rangle \quad \forall\, v, w \in V.$

So for f.d. inner product spaces any linear trans $T : V \to V$ has an adjoint.

Uniqueness of adjoint if it exists

Suppose $T: V \to V$ is linear trans and $S: V \to V$ satisfies $\forall\, v, w \in V$ $\langle T(v), w \rangle = \langle v, S(w) \rangle$.

Claim: S is unique. Suppose $S': V \to V$ is another adjoint. Then $\forall\, v, w \in V$ we have

$$\langle v, S(w) \rangle = \langle T(v), w \rangle = \langle v, S'(w) \rangle$$

$$\Rightarrow \quad \langle v, (S - S')(w) \rangle = 0$$

Since this is true $\forall\, v, w \in V$, take $v = (S - S')(w)$ to conclude $\forall\, w \in V$ that.

$$0 = \langle (S - S')(w), (S - S')(w) \rangle$$

$\therefore \quad (S - S')(w) = 0$ so $S(w) = S'(w)$. Done.

Example where adjoint does not exists.

$V = C^0[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$

V has uncountable basis

$\mathrm{Trig} = \mathrm{Span}\,(1, \cos x, \cos 2x, \ldots; \sin x, \sin 2x, \ldots)$

Recall fact (from Fourier series): $(\mathrm{Trig})^{\perp} = \{0\}$.

The countably linear independent set

$\mathcal{B} = \{1, \cos x, \ldots; \sin x, \sin 2x, \ldots\}$

is not a basis

Complete F to a basis, say

$$\text{Basis} = F \cup D.$$

Define $T: V \to V$

T on F is zero

T on D is Identity.

<u>Claim</u> T has no adjoint.

Suppose $S: V \to V$ is adjoint. Then $\forall v, w \in V$ we have $\langle T(v), w \rangle = \langle v, S(w) \rangle$.

Take $v \in \text{Trig}$. Since T is zero on Trig, we have

$$0 = \langle T(v), w \rangle = \langle v, S(w) \rangle$$

so $S(w)$ must be $\perp$ to Trig. Conclude $S(w) = 0$ $\forall w \in V$

But if S is zero linear trans, then $\langle T(v), w \rangle = 0$ $\forall v, w \in V$. If we take $v \in D$, and $T(v) = v$, and $w = v$ we get contradiction.