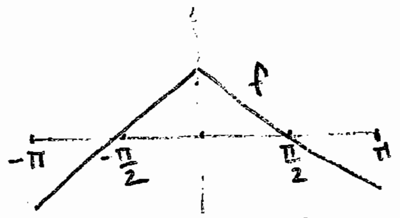
Periodic functions (Fourier Series).

Set P = set of functions continuous functions $f: [-\pi,\pi] \to \mathbb{R}$ so that $f(-\pi) = f(\pi)$.

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx$$ is an inner product on P.

The set $\{1, \cos x, \cos 2x, \dots, \sin x, \sin 2x, \dots\}$ is an orthogonal set. For lengths we have.

$$|1| = \sqrt{2\pi}$$
$$|\cos kx| = \sqrt{\pi}$$
$$|\sin kx| = \sqrt{\pi}$$



Consider the periodic function $f(x) = \begin{cases} -(x-\frac{\pi}{2}) & 0 \le x \le \pi \\ f \text{ is even} \end{cases}$

* Clearly since f even, $\sin kx$ odd, we have $\langle f, \sin kx\rangle = 0$. Also, $\langle f, 1\rangle = 0$

* To compute $\langle f, \cos kx\rangle = 2\int_0^{\pi} -(x-\frac{\pi}{2})\cos kx\,dx$, we integrate by parts. $u = -(x-\frac{\pi}{2})$ $dv = \cos kx\,dx$
$du = -dx$ $v = \frac{\sin kx}{k}$

$$\int_0^{\pi} -(x-\tfrac{\pi}{2})\cos kx\,dx = -(x-\tfrac{\pi}{2})\frac{\sin kx}{k}\Big|_0^{\pi} + \int_0^{\pi}\frac{\sin kx}{k}\,dx$$
$$= 0 + \int_0^{\pi}\frac{\sin kx}{k}\,dx = -\frac{\cos kx}{k^2}\Big|_0^{\pi} = \frac{1}{k^2}(1-\cos \pi k)$$
$$= \begin{cases} 2/k^2 & k \text{ odd.} \\ 0 & k \text{ even} \end{cases}$$

Use computer algebra program such as maxima or maple to graph
$$\frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots + \frac{\cos(2l+1)x}{(2l+1)^2}\right).$$

~~The~~ ~~length~~ square of

$$\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\cdots+\frac{\cos(2l+1)}{(2l+1)^2}\right)$$

is

$$\pi\cdot\left(\frac{4}{\pi}\right)^2\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots+\frac{1}{(2l+1)^2}\right)$$

The length squre of f is $\langle f, f\rangle$

$$2\int_{0}^{\pi}\left(-\left(x-\frac{\pi}{2}\right)\right)^2 dx = \frac{1}{6}\pi^3.$$

Have

$$\frac{1}{6}\pi^3 = \frac{16}{\pi}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$$

$$\frac{\pi^4}{96} = 1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{9^4}+\frac{1}{9^4}+\cdots\ )$$