Simple harmonic oscillator.

Physics differential equation.

Force = mass · acceleration.

Suppose we have a particle of mass 1, and its position along a line is given by $x(t)$.
Assume the force on the particle is negative proportional to its position (displacement) from 0.
Then the physics differential equation $F = ma$ becomes

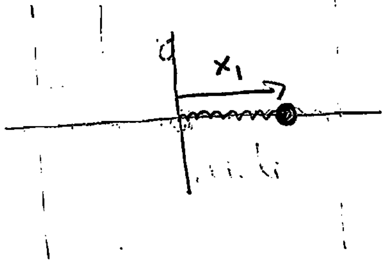
$$x''(t) = -k\,x(t). \quad \text{or} \quad x''(t) + k\,x(t) = 0 \quad (k \text{ positive constant})$$

The set of solutions to $x''(t) + k\,x(t)$ is a vector space.

A basis is given by $x_1(t) = \cos(\sqrt{k}\,t)$
$x_2(t) = \sin(\sqrt{k}\,t)$.

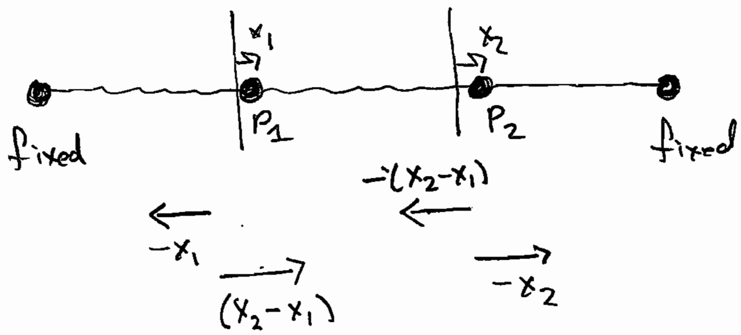
General solution is given by $x(t) = C_1 \cos(\sqrt{k}\,t) + C_2 \sin(\sqrt{k}\,t)$

One is usually given the initial position $x(0)$ and velocity $x'(0)$, from which one can then determine C_1, C_2. Picture of situation is



[illegible] springs which are [illegible] proportional [illegible] far, they are [illegible] modelled.

Multiple simple harmonic oscillator.



Forces on P_1 $\quad -x_1 + (x_2 - x_1)$

Forces on P_2 $\quad -(x_2 - x_1) - x_2$

Differential equation $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}'' = \begin{bmatrix} -2x_1(t) + x_2(t) \\ x_1(t) - 2x_2(t) \end{bmatrix}$

$$= \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Can use the eigenvalues/eigenvectors of the matrix $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ to make a change of variables $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ so the differential equation becomes

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}'' = \begin{bmatrix} -3u_1(t) \\ -u_2(t) \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Characteristic polynomial $\det \begin{bmatrix} x+2 & -1 \\ -1 & x+2 \end{bmatrix} = (x+2)(x+2) - (-1)(-1)$

$$= x^2 + 4x + 4 - 1$$
$$= x^2 + 4x + 3$$
$$= (x+3)(x+1).$$

$E_{-3} := \operatorname{Ker} \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$E_{-1} = \operatorname{Ker} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \operatorname{Span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Take $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, and write T_A for the linear trans $\mathbb{R}^2 \to \mathbb{R}^2$ which is multiplication by $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

Then $$M(T_A)_{\mathcal{E}\to\mathcal{E}} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$M(T_A)_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

$$C_{\mathcal{B}\to\mathcal{E}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$C_{\mathcal{E}\to\mathcal{B}} = C_{\mathcal{B}\to\mathcal{E}}^{-1}$$

$$(C_{\mathcal{B}\to\mathcal{E}})^{-1} M(T_A)_{\mathcal{E}\to\mathcal{E}} \, C_{\mathcal{B}\to\mathcal{E}} = M(T_A)_{\mathcal{B}\to\mathcal{B}}.$$

Take $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_{\mathcal{B}\to\mathcal{E}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

Then $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}'' = C_{\mathcal{B}\to\mathcal{E}} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}''$ so

$$\begin{aligned} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}'' &= C_{\mathcal{B}\to\mathcal{E}}^{-1} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}'' \\ &= C_{\mathcal{B}\to\mathcal{E}}^{-1} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= C_{\mathcal{B}\to\mathcal{E}}^{-1} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} C_{\mathcal{B}\to\mathcal{E}} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &= \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \end{aligned}$$

The equations $\begin{aligned} u_1''(t) &= -3 u_1(t) \\ u_2''(t) &= -1 u_2(t) \end{aligned}$ can be solved as before.