DISPLACEMENT FUNCTIONS ON THE BRUHAT–TITS BUILDING

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ABSTRACT. Suppose G is a semisimple Lie group and K a maximal compact subgroup. Let $\mathcal{B}$ be the symmetric Riemannian space $\mathcal{B} = G/K$. If g an isometry of $\mathcal{B}$, the displacement function d_g is the function on $\mathcal{B}$ which measures the geodesic distance from a point x in $\mathcal{B}$ to the translated point gx. A very striking property of the displacement function d_g is that it is geodesically convex. We show displacement functions for the Bruhat-Tits building also exhibit this convexity property and remark on its use in representation theory.

1. INTRODUCTION

1.1. Let G be a semisimple Lie group and K a maximal compact subgroup. For $g \in G$, the *displacement function* d_g is the function on the symmetric Riemannian space $\mathcal{B} = G/K$ which measure the geodesic distance from a point $x \in \mathcal{B}$ to the translated point gx, that is $d_g(x) := \operatorname{dist}(x, gx)$. Recall that any two distinct points $x_a, x_b \in \mathcal{B}$, can be connected by a unique geodesic path $x(t)$ parametrized by arclength so that $x(0) = x_a$ and $x(c) = x_b$ ($c = \operatorname{dist}(x_a, x_b)$). A function f on $\mathcal{B}$ is geodesically convex if $(1-\alpha)f(x(0)) + \alpha f(x(c)) \geq f(x(\alpha c))$ for all $0 \leq \alpha \leq 1$. A very striking property of the displacement function d_g is that it is geodesically convex [Ms:§5].

Let k denote a nonarchimedean local field of arbitrary characteristic and let G be a connected reductive algebraic group defined over k. Denote by $\mathcal{G}$ the group $G(k)$ of k-rational points of G. We equip $\mathcal{G}$ with the natural locally compact topology induced from that on k. Let $\mathcal{B} = \mathcal{B}(\mathcal{D}G, k)$ be the Bruhat–Tits building of the derived group of G/k. In the nonarchimedean setting, $\mathcal{B}$ is an analogue of the symmetric Riemannian space. Some of the properties shared by both symmetric Riemannian spaces and buildings are

i) They are topologically retractible to a point, i.e. homotopically equivalent to a point.
ii) Given two points in $\mathcal{B}$ (either the symmetric Riemannian space or the building), there is a unique geodesic line segment connecting the two points.
iii) The real and p-adic groups act isometrically on the respective spaces. In the p-adic case, the building $\mathcal{B}$ is a polysimplicial complex and the p-adic group acts ploysimplicially.
iv) In the real case the elements which leave invariant a maximal flat [Ms] is a group whose connected component is a toral group. In the nonarchimedean case the same is true for an apartment in the building.

Set $\mathrm{Aut}(\mathcal{B})$ to be the full group of polysimplicial isometries of $\mathcal{B}$. Our main goal here is to prove the following result.

1.2. Displacement Function Convexity. *Suppose $\mathcal{B} = \mathcal{B}(\mathcal{D}G, k)$. For any $g \in Aut(\mathcal{B})$, the displacement function d_g is a geodesically convex function on $\mathcal{B}$.*

A consequence of the convexity of the displacement function d_g is that for any $r \geq 0$, the set $d_g^{-1}([0,r]) = \{x \in \mathcal{B} | d_g(x) \leq r\}$ is convex. Set

$$d(g) := \inf_{x \in \mathcal{B}} d_g(x).$$

For semisimple Lie groups, Mostow showed [Ms:5.3] that d_g obtains its infinimum if and only if g is a semisimple element. In the nonarchimedean setting we show the infinimum is obtained for any g. The difference in the two situations can be heuristically explained by the observation that any unipotent element of $\mathcal{G}$ is contained in a compact subgroup, while in a semisimple Lie group the identity element is the only unipotent element contained in a compact subgroup.

More generally, if Ξ is any nonempty bounded set of $\mathcal{G}$, motivated by analogies with depth functions and building sets for the Lie algebra introduced in [MP3], we define the function

$$d_{\Xi}(x) := \sup_{h \in \Xi} d_h(x).$$

In particular, $d_{\{g\}} = d_g$. The function d_{Ξ} is convex and we show it attains its infinimum. The set $\mathcal{B}(\Xi)$ where d_{Ξ} achieves its minimum we call the *building set* of Ξ. For any $x \in \mathcal{B}$, the functions d_g allow us to define a filtration of the group $\mathcal{G}$ by compact modulo center sets $\mathcal{G}_{x,r}$ $(r \geq 0)$, which are analogues of the filtrations defined in [MP1,MP2]. Building sets permit us to define a notion of nondegenerate and degenerate cosets in $\mathcal{G}$ and to then prove a descent condition which is analogous to the descent condition established for the Lie algebra and its dual in [MP1,MP2].

We remark that it was the author's desire to formulate a group analogue of the Lie algebra descent condition which motivated investigation into the displacement function. In their proofs of the Howe Conjecture for the Lie algebra, Howe [Ho] and Harish-Chandra [HC] use notions of depth and descent as key tools, but unfortunately neither was able to generalize their approach to prove the group Howe Conjecture. Later, using a very different method, Clozel proved the group version of the Howe Conjecture in [C] for arbitrary reductive groups. It is rather interesting that for SL(2), the displacement function and the descent condition can be used as tools to give a new proof of the group Howe Conjecture along lines very close to Howe's and Harish-Chandra's proof for the Lie algebra. This is done in the final section. The basic idea of the proof can in fact be extended to other groups besides SL(2). We hope to report on the extension to other groups, which is joint work with Barbasch, soon.

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2. Preliminaries on convex functions

2.1. We recall some definitions and facts about continuous convex and concave functions. A subset $Z \subset \mathbb{R}^n$ is *convex* if for any $x, y \in Z$, the line segment $x(t) = tx + (1-t)y$ for $0 \le t \le 1$ is contained in Z. A continuous function $f : Z \longrightarrow \mathbb{R}$ on a convex subset Z is convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \quad \text{for all} \quad 0 \le t \le 1.$$

We shall always assume convex functions are continuous. A function is concave if and only if its negative is convex.

2.2. Proposition. *Let $V = \mathbb{R}^n$ with its usual Euclidean metric, $\|z\| = \sqrt{z \cdot z}$. Suppose $L \in GL(V)$ and $v \in V$. Let g be the affine transformation $gx = g(x) := L(x) + v$. Then the displacement function*

$$d_g(x) := dist(x, gx) = \|x - gx\|$$

is a convex function on V.

Proof. For $x, y \in V$, and $\alpha \ge 0$, $\beta \ge 0$, and $\alpha + \beta = 1$, we have $(\alpha\, gx + \beta\, gy) = \alpha\, L(x) + \alpha\, v + \beta\, L(y) + \beta\, v = L(\alpha\, x + \beta\, y) + v = g(\alpha\, x + \beta\, y)$. Therefore,

$$\begin{aligned} \alpha\, d_g(x) + \beta\, d_g(y) &= \alpha\|x - gx\| + \beta\|y - gy\| \;\ge\; \|\alpha\, x - \alpha\, gx + \beta\, y - \beta\, gy\| \\ &= \|(\alpha\, x + \beta\, y) - (\alpha\, gx + \beta\, gy)\| \;=\; \|(\alpha\, x + \beta\, y) - g(\alpha\, x + \beta\, y)\| \end{aligned}$$

whence d_g is a convex function. □

2.3. The next series of results tell us some special but useful situations when the distance function $d(t) := \mathrm{dist}(p_1(t), p_2(t))$ between two curves $p_1(t)$ and $p_2(t)$ is convex. Roughly, the basic result is that if $p_1(t)$ is a parametrized geodesic line and $p_2(t)$ is a 'bent' line then d is convex.

2.4 Lemma. *Suppose $a < b < c$ are real numbers.*

i) Let $f : [a, c] \longrightarrow \mathbb{R}$ be a continuous function. If f is convex on each of the subintervals $[a, b]$ and $[b, c]$, and f is also convex in a neighborhood of b, then f is convex on the entire interval $[a, c]$.

ii) Let $f_1 : [a,b] \longrightarrow \mathbb{R}$ *and* $f_2 : [b,c] \longrightarrow \mathbb{R}$ *be convex differentiable functions so that* $f_1(b) = f_2(b)$*, and* $f_1'(b) = f_2'(b)$*. Then the function* f *obtained by splicing* f_1 *and* f_2 *together, i.e.*

$$f(t) = \begin{cases} f_1(t) & \textit{for } a \le t \le b \\ f_2(t) & \textit{for } b \le t \le c. \end{cases}$$

is convex on the interval $[a,c]$*.*

2.5 Proposition. *Suppose* $v_1, v_2 \in V = \mathbb{R}^n$*. For any* $x_1, x_2 \in V$*, set*

$$p_1(t) := x_1 + tv_1 \quad \textit{and} \quad p_2(t) := x_2 + tv_2 \tag{2.5.1}$$

i) The function

$$f(t) := \mathit{dist}(p_1(t), p_2(t))$$

is a convex function on $\mathbb{R}$*.*

ii) Let H *be any affine hyperplane in* V *which contains* x_2 *and denote by* s *the reflection across the hyperplane* H*. Set*

$$p_{2'}(t) := \begin{cases} p_2(t) & t \le 0 \\ s(p_2(t)) & t \ge 0 \end{cases} \tag{2.5.2}$$

Suppose either

(1) $x_1 \in H$*, or*
(2) *For small positive values of* t*, the points* $p_1(t)$ *and* $p_2(t)$ *are in the same half space of* H *in* V*.*

Then the function

$$h(t) := \mathit{dist}(p_1(t), p_{2'}(t))$$

is a convex function on $\mathbb{R}$*.*

Proof. The proof of assertion i) is an easy corollary to Proposition 2.2. Indeed, we can find an affine transformation $g : V \longrightarrow V$ so that $g(p_1(t)) = p_2(t)$ and thus $f(t) = \mathrm{dist}(p_1(t), p_2(t)) = d_g(p_1(t))$ is convex.

To prove ii), we first remark that by i), the function h is convex on each of the intervals $(-\infty, 0]$ and $[0, +\infty)$. To prove part (1), we consider two cases depending the on whether x_1 and x_2 are equal.

Case1: $x_1 = x_2$. Here, the function h has a (global) minimum value of 0 at $t = 0$. This and convexity of h on each of the intervals $(-\infty, 0]$ and $[0, +\infty)$ implies h is convex.

Case 2: $x_1 \neq x_2$. The function $k(t) = \mathrm{dist}(s(p_1(t)), s(p_2(t)))$ is obviously a convex function and $k(0) = \mathrm{dist}(x_1, x_2) = f(0)$. Since $x_1 \neq x_2$, the functions f and k are smooth in a neighborhood of $t = 0$. We can resolve the differences $p_2(t) - p_1(t)$ and $s(p_2(t)) - p_1(t)$ into vectors parallel and perpendicular to H to obtain

$$p_2(t) - p_1(t) = \{(x_2 - x_1) + tv_h\} \ + \ tv_a$$
$$s(p_2(t)) - p_1(t) = \{(x_2 - x_1) + tv_h\} \ + \ tv_b$$

The component $u(t) := \{(x_2 - x_1) + tv_h\}$ is parallel to H, is affine and $u(0) \neq 0$. The vectors v_a and v_b are perpendicular to H. An elementary calculation shows that f and k are differentiable at the origin and $f'(0) = k'(0)$. By Proposition 2.3, the 'spliced' function which is equal to $f(t)$ for $t \leq 0$ and $k(t)$ for $t \geq 0$ is convex. This spliced function is precisely $\mathrm{dist}(p_1(t), p_{2'}(t)) = h(t)$. This completes the proof of (1).

To prove part (2), we calculate that for $t \geq 0$ $\mathrm{dist}(p_1(t), sp_2(t)) \geq \mathrm{dist}(p_1(t), p_2(t))$. We conclude from this that h is convex in a neighborhood of $t = 0$. Since we also know h is convex in each of the intervals $t \leq 0$ and $t \geq 0$, it must in fact be convex. □

3. The retraction map and the Bruhat decomposition

3.1. We recall some notation and facts from [MP2] and [BT2]. Let k denote a nonarchimedean local field of arbitrary characteristic and let G be a connected reductive algebraic group defined over k. Let K be a fixed maximal unramified extension of k and $\Gamma = \mathrm{Gal}(K/k)$. Let $\mathfrak{o}$ and $\mathfrak{O}$ denote the ring of integers of k and K respectively, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ and let $\mathfrak{f}$ (resp. $\mathfrak{F}$) be the residue field of k (resp. K). Let $\mathcal{D}G$ be the derived group of G. The Bruhat–Tits buildings $\mathcal{B}(\mathcal{D}G, K)$ and $\mathcal{B}(\mathcal{D}G, k)$ of $\mathcal{D}G/K$ and $\mathcal{D}G/k$ respectively are canonically defined. The (*enlarged*) Bruhat–Tits building $\mathcal{B}(G, K)$ (resp. $\mathcal{B}(G, k)$) of G/K (resp. of G/k or of $\mathcal{G}$) is the product of $\mathcal{B}(\mathcal{D}G, K)$ (resp. $\mathcal{B}(\mathcal{D}G, k)$) with $X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$; where S is the maximal K-split (resp. k-split) torus contained in the center of G. There is a natural action of the Galois group Γ on $\mathcal{B}(G, K)$, and $\mathcal{B}(G, k)$ can (and will) be identified with the subset of points of $\mathcal{B}(G, K)$ fixed under Γ.

To each point x of the building $\mathcal{B}(G, k)$ of G/k, Bruhat and Tits [BT2] have associated a subgroup of $G(K)$, the *parahoric subgroup* determined by x, to be denoted by G_x, which is a certain subgroup of finite index in the isotropy group at x. (The parahoric subgroup is defined to be the inverse image in the isotropy group at x of the identity component of reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated with the isotropy subgroup. If G is a semi-simple simply connected group, then the parahoric subgroup coincides with the isotropy subgroup.) The parahoric subgroup is "defined" over k i.e. it is stable under Γ; the subgroup $\mathcal{G} \cap G_x$ will be denoted by $\mathcal{G}_x$ and it is by definition the parahoric subgroup of $\mathcal{G}$ determined by (or associated with) the point x; it is a compact-open subgroup of $\mathcal{G}$. A minimal parahoric subgroup is called an *Iwahori subgroup*. It is known that the Iwahori subgroups are conjugate to each other under $\mathcal{G}$.

The buildings $\mathcal{B}(\mathcal{D}G, k) \subset \mathcal{B}(\mathcal{D}G, K)$ are polysimplicial complexes. If F is a facet in $\mathcal{B}(\mathcal{D}G, K)$, denote by G_F the parahoric subgroup of $G(K)$ fixing the facet F. Recall that a point $x \in F$ is an interior point if x does not lie in any facet $E \subsetneq F$. It is the case that $G_F = G_x$ for any interior point x in F. If the facet F lies in $\mathcal{B}(\mathcal{D}G, k)$, $\mathcal{G}_F = G_F \cap \mathcal{B}$ is the maximal parahoric subgroup of $\mathcal{G}$ which fixes F.

3.2. Let S be a maximal k-split torus of G and let $A = A(S)$ be the apartment in $\mathcal{B}$ associated to the torus S. Let Z and N denote respectively the centralizer S and normalizer of S in G. Since S is defined over k, so are Z and N. If F is a facet in A, define

$$W_F(S) := (N(K) \cap \mathcal{G}_F)/(Z(K) \cap \mathcal{G}_F).$$

Then $W_F(S)$ is naturally isomorphic to the Coxeter group generated by the reflections in A of affine root hyperplanes containing the facet F.

3.3. Denote by $G_{F,+}$ the maximal normal pro-p-subgroup of G_F. The quotient group $G_F/G_{F,+}$ can be identified with the group of $\mathfrak{F}$-rational points of the maximal reductive quotient M_F of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated to the parahoric subgroup G_F. The Galois group Γ acts on the groups G_F, $G_{F,+}$ and M_F with fixed points $\mathcal{G}_F$, $\mathcal{G}_{F,+}$ and $\mathcal{G}_F/\mathcal{G}_{F,+} \cong \mathsf{M}_F(\mathfrak{f})$. Let $S(K)_c$ be the maximal bounded subgroup of $S(K)$. The image of $S(K)_c$ in $\mathsf{M}_F(\mathfrak{F})$ is the $\mathfrak{F}$-rational points of a maximal split $\mathfrak{f}$-torus $\mathsf{S}_F \subset \mathsf{M}_F$. The $\mathfrak{f}$-rational points of the $\mathfrak{f}$-torus S is canonically the image of $S(K)_c \cap \mathcal{G}_F$.

3.4. Given two facets there is an apartment containing them both. Let $A = A(S)$ be an apartment and $C \subset A$ a chamber. For every apartment A' containing C there is a unique isometry $\phi_{A'} : A' \longrightarrow A$ which fixes C. All these isometries patch together in a coherent manner to yield a mapping $\rho_{A,C} : \mathcal{B} \longrightarrow A$ called the *retraction* of $\mathcal{B}$ onto A with center C. Clearly, geodesic segments are mapped by $\rho_{A,C}$ onto broken lines, i.e. connected union of finitely many geodesic segments.

3.5. Proposition. *(BT1:Proposition 7.4.20) Suppose C is a chamber in the apartment A. Set $\rho = \rho_{A,C}$. Then for all $y, z \in \mathcal{B}$*

$$\mathit{dist}(\rho(y), \rho(z)) \leq \mathit{dist}(y, z).$$

Suppose $A = A(S)$ is an apartment in the building $\mathcal{B}$. If ψ is an affine root with respect to S, set

$$H_\psi := \{x | \psi(x) = 0\}. \tag{3.5.1}$$

If $\mathcal{Y}$ is a nonempty subset of A, set

$$\mathcal{C}(\mathcal{Y}) := \{x \in A \mid \psi(x) \geq 0 \text{ whenever } \psi \text{ is an affine root such that } \psi|_{\mathcal{Y}} \geq 0\}. \tag{3.5.2}$$

The set $\mathcal{C}(\mathcal{Y})$ is the smallest convex set which contains $\mathcal{Y}$ and is a union of closed affine Weyl facets of A. We call $\mathcal{C}(\mathcal{Y})$ the *convex chamber closure* of $\mathcal{Y}$. In particular, if $\mathcal{Y}$ contains an open subset of A, then $\mathcal{C}(\mathcal{Y})$ is a union of chambers.

3.6. Lemma. *Let C be a chamber and F a facet in an apartment $A = A(S)$. Let $\mathcal{C}(C \cup F)$ be the convex chamber closure of C and F. The facet F lies in a unique chamber of $\mathcal{C}(C \cup F)$.*

Proof. Suppose to the contrary that F is a facet for two different chambers C_a and C_b in $\mathcal{C}(C \cup F)$. Let H_ψ be a root hyperplane in A which separates the two chambers. Relabel the chambers so that C_a and C lie in the same half space B of H_ψ. Then $\mathcal{C}(C \cup F) \subset B$, a contradiction to $C_b \subset \mathcal{C}(C \cup F)$. $\square$

3.7. Let

$$C(C, F) := \text{ the unique chamber in } \mathcal{C}(C \cup F) \text{ containing } F. \tag{3.7.1}$$

Also, let $\operatorname{star}(F)$ be set of all facets containing the facet F. Each facet $E \neq F$ in $\operatorname{star}(F)$ corresponds to a parahoric subgroup $\mathcal{G}_E \subsetneq \mathcal{G}_F$ and hence determines a parabolic subgroup $\mathcal{G}_E/\mathcal{G}_{F,+}$ in $\mathcal{G}_F/\mathcal{G}_{F,+} \cong \mathsf{M}_F(\mathfrak{f})$. Equivalently, each facet $E \neq F$ determines a facet in the spherical Tits building $\mathcal{T} = \mathcal{T}(\mathsf{M}_F(\mathfrak{f}))$ of $\mathsf{M}_F(\mathfrak{f})$. Denote by $P(E)$ the parabolic subgroup $\mathcal{G}_E/\mathcal{G}_{F,+}$ of $\mathcal{G}_F/\mathcal{G}_{F,+}$. The topological subspace $\operatorname{Star}(F)$ of $\mathcal{B}$ which is the union of the facets of $\operatorname{star}(F)$ is polysimplicially equivalent to the suspension of the spherical Tits building $\mathcal{T}$ by the facet F, i.e. the topological join of $\mathcal{T}$ and F. The space $\operatorname{Star}(F)\backslash F$ is homotopically equivalent to $\mathcal{T}$. The maximal split $\mathfrak{f}$-torus S associated to S determines an apartment $\mathcal{A}$ in $\mathcal{T}$. The topological suspension of $\mathcal{A}$ by F is naturally simpilicially equivalent to $A(S) \cap \operatorname{Star}(F)$. We can and do use the chamber $C_1 = C(C, F)$ in (3.7.1) as a base chamber to obtain a bijection of the chambers in the apartment $\mathcal{A}$ with elements of the Weyl group $W = W_F(S)$ of S in $\mathcal{G}_F/\mathcal{G}_{F,+}$. If $w \in W$, let $\tilde{w} \in N(S) \cap \mathcal{G}_F$ be an element which maps to $w \in W$, then the bijection is

$$w \longleftrightarrow \tilde{w} C_1. \tag{3.7.2}$$

When the context is clear, for notational convenience, we shall sometimes identify the chamber with its corresponding Weyl element. Denote the length of an element $w \in W$ by $\ell(w)$. Let $\succeq$ (and $\succ$) denote the Bruhat ordering [Hp:5.9] on the elements of $W_F(S)$ and let w_0 be the maximal element. We have $\ell(w_0) \geq \ell(w)$. The chamber $\tilde{w_2}\tilde{w_0}C_1$ is the chamber opposite the chamber $\tilde{w_2}C_1$.

We recall the following result on the Bruhat ordering.

3.8. Proposition. *([Hp:5.8,5.10,5.11]) Suppose V is an Euclidean space and W the finite Weyl group generated by a root system in V.*

i) Let $e \in W$ be the identity element and suppose $s \in W$ is a reflection across a root hyperplane H in V. For any $w \in W$, a necessary and sufficient condition for the Weyl chambers C_e and C_w to lie in the same half space of H in V is that $w \prec sw$.

ii) Suppose $w_a \prec w_b$. Then there is a sequence of root hyperplanes $H_1, H_2, \ldots, H_n$ with corresponding reflections $s_1, s_2, \ldots, s_n$ so that $w_b = s_n \cdots s_1 w_a$ and for $i = 1, \ldots, n$

$$s_{i-1} \cdots s_1 w_a \;\prec\; s_i \cdots s_1 w_a. \tag{3.8.1}$$

We remark that the proposition can also be formulated with the reflections on the right instead of the left side.

3.9. The chambers of $\mathcal{T} = \mathcal{T}(\mathsf{M}_F(\mathfrak{f}))$ are in bijection with the points of the flag variety $\mathsf{M}_F(\mathfrak{f})/P(C_1)$ of $\mathfrak{f}$-rational Borel subgroups of M_F. The Bruhat decomposition says the $P(C_1)$ orbits of the flag variety $\mathsf{M}_F(\mathfrak{f})/P(C_1)$ are parametrized by the Weyl group $W_F(S)$. A $\mathfrak{f}$-rational Borel subgroup $B \in \mathsf{M}_F(\mathfrak{f})/P(C_1)$ is in relative position $w \in W_F(S)$ to $P(C_1)$ if B lies in the $P(C_1)$-orbit $P(C_1)\tilde{w}B$, where $\tilde{w}$ is an element of the normalizer $\mathsf{N}(\mathsf{S})$ which reduces to w. A Borel subgroup B is opposite to $P(C_1)$, if it is relative position w_0 to $P(C_1)$. Equivalently, B is opposite to $P(C_1)$ if their intersection $B \cap P(C_1)$ is the $\mathfrak{f}$-rational points of a maximal $\mathfrak{f}$-torus.

3.10. Proposition. *Let $A = A(S)$ be an apartment in $\mathcal{B}(G)$ and C a chamber in A. Suppose F is a facet in A and $C_1 = C(C,F)$ is the chamber defined in (3.7.1). Let $\rho = \rho_{A,C}$ be the retraction of $\mathcal{B}$ to A centered at C. Then for D a chamber in star(F),*

$$\textit{position of } D \textit{ relative to } C_1 = \textit{ position of } \rho_{A,C}(D) \textit{ relative to } C_1.$$

Proof. The retraction ρ fixes A pointwise. There exists an element $g \in \mathcal{G}$ so that $\rho(x) = gx$ for all $x \in D \cup C_1$; in particular, g must fix C_1 pointwise. Hence g lies in the Iwahori subgroup $\mathcal{G}_{C_1}$. In $\mathcal{G}_F/\mathcal{G}_{F,+}$, consider the three Borel subgroups $P(C_1)$, $P(D)$ and $P(\rho_{A,C}(D))$. The image $\overline{g}$ of g in $\mathcal{G}_F/\mathcal{G}_{F,+}$ lies in the Borel subgroup $P(C_1)$ and conjugates $P(D)$ to $P(\rho_{A,C}(D))$. Since conjugation of a Borel subgroup by an element of $P(C_1)$ does not change its relative position with respect to $P(C_1)$, the assertion is proved. $\square$

3.11. Corollary. *Given a chamber $C \subset A = A(S)$ and a facet $F \subset C$, suppose $D \in$ star(F) is a chamber opposite to C, then the chamber $\rho_{A,C}(D)$ is opposite C.*

3.12. We remark that if F is a facet in A, then Proposition 3.10 completely determines the maping $\rho_{A,C}$ on the subset Star(F). Indeed, define the chamber C_1 as in (3.7.1). If D is a chamber in star(F), express the Borel subgroup $P(D)$ as $P(D) = Ad(gn)P(C_1)$, where $g \in P(C_1)$ and $n \in N(\mathsf{S})$. Choose representatives $\tilde{g} \in \mathcal{G}_{C_1}$ and $\tilde{n} \in N(S) \cap \mathcal{G}_F$ for g and n respectively. Then

$$\rho_{A,C}(x) = \tilde{g}^{-1}x \text{ for every } x \in D. \tag{3.12.1}$$

3.13. Suppose $D_1, D_2 \in$ star(F) are chambers so that the Borel subgroups $P(D_1)$ and $P(D_2)$ are in opposite positions. Let $w_1, w_2 \in W_F(S)$ be the relative positions of $P(D_1)$ and $P(D_2)$ with respect to a Borel subgroup $P(C)$. The next result tells us the relationship between w_1 and w_2.

3.14. Proposition. *Let $D_1 \in$ star(F) be a chamber so that the Borel $P(D_1)$ is in relative position w_1 to the Borel $P(C)$. A necessary and sufficient condition for there to be a Borel $P(D_2)$ opposite to $P(D_1)$ and in relative position w_2 to $P(C)$ is that $w_1 \succeq w_2 w_0$.*

Proof. Denote $P(C)$ as B. A pair of Borel subgroups (xB, yB) $(x, y \in \mathsf{M}_F(\mathfrak{f})/B)$ is in relative position w if and only if $x^{-1}y \in BwB$. To say $P(D_1)$ and $P(D_2)$ are in relative positions w_1 and w_2 to B means that $P(D_1) = Ad(g_1)B$ and $P(D_2) = Ad(g_2)B$, where $g_1 \in Bw_1B$ and $g_2 \in Bw_2B$ respectively. To say the two Borels $P(D_1)$ and $P(D_2)$ are in relative position w_0 means $g_2^{-1}g_1 \in Bw_0B$ and therefore $Bw_0B \subset Bw_2^{-1}Bw_1B$. This is equivalent to the stated condition. □

We remark that the roles of D_1 and D_2 are symmetrical; $w_1 \succeq w_2w_0$ is equivalent to $w_2 \succeq w_1w_0$

4. Proof of Displacement Function Convexity

4.1. We give in this section the proof of our main theorem that a displacement function d_g is a convex function on the building. The basic idea of the proof is to calculate the displacement function along a geodesic path $v(t)$. Suppose the geodesic path passes through a chamber C. Let I be the geodesic segment of $v(t)$ in C. To calculate the displacement function along I, take A be an apartment which contains both C and gC. The map $g : C \longrightarrow gC$ extends to an isometry $\tilde{g}$ of A. The displacement function $d_{\tilde{g}}$ of $\tilde{g}$ on A is equal to d_g on the chamber C. Since $d_{\tilde{g}}$ is convex by Proposition 2.2, we conclude d_g is convex on C and hence the segment I. Therefore, as a consequence of Lemma 2.4, to prove a displacement d_g is convex, it is sufficient to establish it is convex along any geodesic segment consisting of the intersection of $v(t)$ with two adjacent chambers.

4.2. Suppose F is a facet in $\mathcal{B}$ and C_r and C_s are two chambers containing F so that $P(C_r)$ and $P(C_s)$ are opposite Borel subgroups in $\mathsf{M}_F(\mathfrak{f})$. Furthermore, let x_r and x_s be points in the interior of C_r and C_s respectively with the property that the geodesic $x(t)$ from x_r to x_s meets the facet F, say $x(0) = x_r$, $x(b) \in F$ and $x(c) = x_s$ where $c = \mathrm{dist}(x_r, x_s)$. The value b is unique since x_r and x_s are interior points. Let $I = x((b-\epsilon, b+\epsilon))$ be a small neighborhood of the point $x(b)$ in the geodesic x. Our goal is to show that the function $h(t) = \mathrm{dist}(x(t), gx(t))$ is convex in a neighborhood of $t = b$. Convexity in a neighborhood of $t = b$ is obvious if there exists an apartment A which contains both I and gI. Unfortunately, I and gI may not lie in a single apartment. We shall show there is an apartment A containing C_r so that the retraction map $\rho = \rho_{A,C_r}$ retracts x to a geodesic in A, and gx to a bent geodesic in A with the property that $f(t) = \mathrm{dist}(\rho(x(t)), \rho(gx(t)))$ is convex; and since $h(t) \geq f(t)$, we deduce that h is convex.

For any apartment A containing the chamber C_r, let $\rho = \rho_{A,C_r}$ be the retraction map of the building $\mathcal{B}$ to the apartment A centered at the chamber C_r. We claim the retraction of $x(t)$, i.e.

$$v(t) := \rho(x(t)) \tag{4.2.1}$$

is a geodesic line segment in A. Indeed, if A' is any apartment containing C_r and C_s, the retraction map is the unique isometry of A' to A which fixes C_r pointwise. Note also that by Corollary 3.11, $P(C_r)$ and $P(\rho_{A,C_r}(C_s))$ are opposite Borel subgroups in $\mathsf{M}_F(\mathfrak{f})$.

Suppose $g \in \mathcal{G}$. By Lemma 3.6, the convex set $\mathcal{C}(C_r \cup gF)$ contains a unique chamber $C = C(C_r, gF)$ incident with the facet gF. Let A be an apartment containing the chambers C_r and $C'_r := gC_r$ and let $\rho = \rho_{A,C_r}$ be the retraction map of the building $\mathcal{B}$ to the apartment A centered at the chamber C_r. Set $C'_s = gC_s$. Both C'_r and C'_s belong to the set $\mathrm{star}(gF)$. Proposition 3.10 tells us that the retraction map on chambers in $\mathrm{star}(gF)$ (hence on all facets in $\mathrm{star}(F)$) is completely determined by the relative positions of Borel subgroups in $\mathsf{M}_{gF}(\mathfrak{f})$ to $P(C)$. Define the path $z(t)$ in A as

$$z(t) := \rho(gx(t)), \text{ for } 0 \leq t \leq c. \tag{4.2.2}$$

Set $z_r := z(0)$ and $z_s := z(c)$. The path $z(t)$ is a 'bent' geodesic path. It is a geodesic line segment in the chamber C'_r for $0 \leq t \leq b$, and in the chamber $\rho(C'_s)$ for $b \leq t \leq c$.

4.3. Proposition. *For $v(t)$ and $z(t)$ as in (4.2.1) and (4.2.2), the function*

$$f(t) := dist(v(t), z(t)) \tag{4.3.1}$$

is a convex function.

Proof. We have $z(0) = \rho(gx_r) \in \rho(C'_r)$, $z(b) = \rho(gx(b)) = gx(b) \in gF$, and $z(c) = \rho(gx_s) \in \rho(C'_s)$. Let $z_1(t)$ be the geodesic line in the apartment A parametrized by arclength which begins at $z(0)$, i.e. when $t = 0$ and is at $z(b)$, when $t = b$; and let $z_2(t)$ be the geodesic line which is at $z(b)$ when $t = b$, and at $z(c)$ when $t = c$. Each of the functions

$$f_1(t) := \mathrm{dist}(v(t), z_1(t)) \quad \text{and} \quad f_2(t) := \mathrm{dist}(v(t), z_2(t))$$

is convex. The function f is obtained by 'splicing' f_1 on the interval $[0, b]$ to f_2 on the interval $[b, c]$. To show f is convex, it suffices to show it is convex in a neighborhood of the point b. Write the retractions $\rho(C'_r)$ and $\rho(C'_s)$ as $\rho(C'_r) = w_1C$ and $\rho(C'_s) = w_2C$. The chamber w_2C contains $z_2(c)$, and the chamber $D := w_1w_0C$ (in A) opposite to w_1C contains $z_1(c)$. Since C'_r and C'_s are opposite chambers with respect to the facet gF, Proposition 3.14 says $w_1w_0 \preceq w_2$. If $w_1w_0 = w_2$, then $z(t) = \rho(g(x(t)))$ is a geodesic line in A; in particular, $f(t)$ is obviously convex. We may therefore assume $w_1w_0 \prec w_2$. By Proposition 3.8, there is a sequence of root hyperplanes $H_1, H_2, \ldots, H_n$ with corresponding reflections $s_1, s_2, \ldots, s_n$ so that $w_2 = s_n \cdots s_1w_1w_0$ and $s_{i-1} \cdots s_1w_1w_0 \prec s_i \cdots s_1w_1w_0$ for $i = 1, \ldots, n$. Lift the reflections s_i to elements $n_i \in N(S) \cap \mathcal{G}_{gF}$. For $i = 1, \ldots, n+1$, define

$$u_i(t) := \begin{cases} z_1(t) & 0 \leq t \leq b \\ n_{i-1} \cdots n_1z_1(t) & b \leq t \leq c. \end{cases}$$

Then, $u_1(t) = z_1(t)$ and $u_{n+1}(t) = z(t)$. We shall prove by induction on i that the functions

$$d_i(t) := \mathrm{dist}(v(t), u_i(t))$$

are convex functions. For $i = 0$ this is obvious since $u_1(t) = z_1(t)$ is a geodesic line segment in A. Assume d_{i-1} is a convex function. The facet gF lies on all the hyperplanes H_i, and so $u_i(b) = z(b)$ for all i. The action of n_i on A takes the line segment $[z(b), u_{i-1}(c)]$ to the line segment $[z(b), u_i(c)]$. We consider two cases depending whether or not the facet F lies in H_i or in the interior of one of the two half spaces determined by H_i.

Case 1. F lies in H_i. We apply Proposition 2.5 to conclude that if the function d_{i-1} is convex on the interval $[0, c]$, then so is the function d_i.

Case 2. F does not lie in H_i. Here, the chambers C_r, $\rho(C_s)$, $C'_r = gC_r$ and $n_{i-1} \cdots n_1 D$ all lie in the same half space of H_i. Hence, it is obvious, using the orthogonal coordinate system provided by H_i and a perpendicular line to H_i that $\mathrm{dist}(v(t), u_{i-1}(t)) \leq \mathrm{dist}(v(t), u_i(t))$ for $b \leq t \leq c$, and therefore d_i is a convex function. □

4.4. Displacement Function Convexity Theorem. *Suppose $g \in Aut(\mathcal{B}(\mathcal{D}G, k))$. The displacement function $d_g : \mathcal{B} \longrightarrow \mathbb{R}$ defined as $d_g(x) := dist(x, gx)$ is a convex function.*

Proof. As already mentioned in section 4.1, the restriction of d_g to any chamber $C \subset \mathcal{B}$ is a convex function and that global convexity will follow if we can establish convexity on the union $C_a \cup C_b$ of two adjacent chambers. In particular, it is enough to show that if $x(t)$ is a geodesic segment which begins in the chamber C_a and ends in the adjacent chamber C_b, then the function

$$h(t) := \mathrm{dist}(x(t), gx(t))$$

is a convex function. Furthermore, as it is the case that if a sequence of convex functions converges pointwise, then the limit function is also convex, we need only consider the case where the starting and ending points are interior points. So, suppose $x(t)$ $(0 \leq t \leq c)$ is a geodesic segment which starts at a point $x_a = x(0) \in C_a$ and ends at a point $x_b = x(c) \in C_b$ and passes through the facet $F = C_a \cap C_b$, say $x(\beta) \in F$. Let A be an apartment containing the chambers C_a and gC_a. Let $\rho = \rho_{A,C_a}$ be the the retraction of the building $\mathcal{B}$ to the apartment A centered at the chamber C_a. The retraction ρ takes the union $C_a \cup C_b$ isometrically to $C_a \cup \rho(C_b)$; whence, as was already mentioned before (4.2.1), the retraction $\rho(x(t))$ of $x(t)$ is a geodesic segment in A between the two chambers C_a and $\rho(C_b)$. We have by Proposition 3.5, that $h(t) = \mathrm{dist}(x(t), gx(t)) \geq \mathrm{dist}(\rho(x(t)), \rho(gx(t)))$. Set $f(t) := \mathrm{dist}(\rho(x(t)), \rho(gx(t)))$. We have $h(t) = f(t)$ for $0 \leq t \leq \beta$. Since the function h is convex on each of the intervals $0 \leq t \leq \beta$ and $\beta \leq t \leq c$, it follows that a sufficient condition for h to be convex is that function f is convex. Hence it is enough to show f is convex. But this is precisely the situation of Proposition 4.3. □

5. Filtrations sets and building sets

5.1. Let $x \in \mathcal{B} = \mathcal{B}(\mathcal{D}G, k)$ and $r \geq 0$. Set

$$\mathcal{G}_{x,r} := \{\ g \in \mathcal{G} \mid d_g(x) \leq r\ \}. \tag{5.1.1}$$

The set $\mathcal{G}_{x,r}$ is the elements in $\mathcal{G}$ which move the point x a distance at most r. It is thus obvious that $\mathcal{G}_{x,0}$ is the subgroup $\mathrm{Stab}(x)$ of elements in $\mathcal{G}$ which fix the point x; in particular, $\mathcal{G}_{x,0}$ contains the parahoric subgroup $\mathcal{G}_x$ which fixes x. We make some remarks. The first is that for $r > 0$, the sets $\mathcal{G}_{x,r}$ are not subgroups. In particular, they should not be confused with the groups denoted by the same symbol in [MP2]. A definition more compatible with the use of the symbol $\mathcal{G}_{x,r}$ in [MP1] is to replace the definition in (5.1.1) with $\mathcal{G}_{x,-r} := \{ \, g \in \mathcal{G} \mid d_g(x) \leq r \, \}$. Since this would be somewhat cumbersome in the present setting, we use here the definition (5.1.1) as stated. The sets $\mathcal{G}_{x,r}$ are closed under right multiplication by the group $\mathcal{G}_{x,0}$. If $0 \leq r_1 \leq r_2$, then $\mathcal{G}_{x,r_1} \subset \mathcal{G}_{x,r_2}$ and the union of the filtration sets exhaust $\mathcal{G}$, i.e.

$$\mathcal{G} = \bigcup_{r \geq 0} \mathcal{G}_{x,r}.$$

The second remark is that if the center of G/k is not anisotropic over k, then the group $\mathcal{G}_{x,0}$ is not compact. The noncompactness is precisely because the k-rational points of the center are contained in $\mathcal{G}_{x,0}$. In this section we shall assume the center of G/k is anisotropic and therefore $\mathcal{G}_{x,r}$ are compact open subsets of $\mathcal{G}$ for all $r \geq 0$. The third remark is that a reader familar with the notation in [MP1,MP2,MP3] may extrapolate the notation there and believe $\mathcal{G}_{x,0}$ is the parahoric subgroup $\mathcal{G}_x$. In our current situation, $\mathcal{G}_{x,0}$ and $\mathcal{G}_x$ are different groups; $\mathcal{G}_{x,0}$ is the stability subgroup of x while $\mathcal{G}_x$ is the parahoric subgroup for x.

5.2. For any $g \in \mathcal{G}$, it is obvious that have $d_h(x) = d_g(x)$ for all $h \in g\mathcal{G}_{x,0}$. Futhermore for any $R \geq 0$, there are only finitely many cosets $g\mathcal{G}_{x,0}$ so that $d_{g\mathcal{G}_{x,0}}(x) \leq R$. Therefore, the set of values $g \mapsto d_g(x)$ is a discrete set in $\mathbb{R}$ with no accumulation points. For $r > 0$, define

$$\mathcal{G}_{x,r^-} := \bigcup_{0 \leq s < r} \mathcal{G}_{x,s}.$$

As an immediate consequence of the remarks before the definition we have

5.3. Proposition. *Suppose $x \in \mathcal{B}$.*

i) If $r \geq 0$, there exists an $\epsilon > 0$ depending on x and r such that $\mathcal{G}_{x,r} = \mathcal{G}_{x,s}$ for every $r \leq s < r + \epsilon$.

ii) If $r > 0$, there exists a $\delta > 0$ depending on x and r such that $\mathcal{G}_{x,r^-} = \mathcal{G}_{x,s}$ for every $r - \delta < s < r$

5.4. If Ξ is a bounded nonempty subset of $\mathcal{G}$, set

$$d_\Xi(x) := \sup\{ \, d_g(x) \mid g \in \Xi \, \}. \tag{5.4.1}$$

Since the supremum of a set of convex functions is convex, the function d_Ξ is a convex function on $\mathcal{B}$. Set

$$d(\Xi) := \inf_{x \in \mathcal{B}} d_\Xi(x). \tag{5.4.2}$$

For $g \in \mathcal{G}$ shorten $d_{\{g\}}$ and $d(\{g\})$ to d_g and $d(g)$. The next results says that the function d_g achieves $d(g)$ as its minimal value.

5.5. Proposition. *Given an nonempty bounded subset Ξ of $\mathcal{G}$, there is a point $x \in \mathcal{B}$ so that*

$$d_{\Xi}(x) = d(\Xi).$$

Proof. We first prove the result when Ξ is a single element g. Choose any point $z \in \mathcal{B}$. View [BT1] the set of geodesic rays in $\mathcal{B}$ emmanating from the point z as the compact spherical building $\mathcal{T} = \mathcal{T}(\mathcal{D}G, k)$ of $\mathcal{D}G/k$. Define the continuous function $m : \mathcal{T} \longrightarrow \mathbb{R}$ to be

$$m_g(r) := \text{ infinimum of the function } d_g \text{ on the ray } r.$$

Since $\mathcal{T}$ is compact, there is a ray $r(t)$, $(t \geq 0)$, i.e. a point in $\mathcal{T}$, where the function m achieves its minimum. Clearly $m_g(r) = d(g)$. There are two possibilities. Either d_g achieves its minimum on the ray r, or $\lim_{t\to\infty} d_g(r(t)) = d(g)$.

Case 1: d_g achieves its minimum on the ray r. Suppose the minimum is achieved for $t = t_1$. But then $d_g(r(t_1)) = d(g)$, i.e. d_g achieves its minimum at the point $x_1 = r(t_1)$.

Case 2: $\lim_{t\to\infty} d_g(r(t)) = d(g)$. Here, necessarily d_g must decrease along the ray r. The transform gr of the geodesic ray by g is another geodesic ray. By [BT1] for $t_0 \geq 0$ sufficiently large the two geodesic rays $r([t_0, \infty))$ and $gr([t_0, \infty))$ both lie in an apartment A and hence are rays in the apartment. It is the case that

i) The function $t \mapsto \operatorname{dist}(r(t), gr(t))$ measures the Euclidean distance between points on the parametrize rays r and gr.
ii) $\lim_{t\to\infty} \operatorname{dist}(r(t), gr(t)) = d(g)$.

It is easy to see that the form of the Euclidean distance function forces the existence of an t_1 so that $\operatorname{dist}(r(t_1), gr(t_1)) = d(g)$. Hence, the function d_g achieves its minimal value at the point $x_1 = r(t_1)$. This completes the proof of the proposition when $\Xi = \{g\}$.

Suppose now Ξ is an arbitrary nonempty bounded set. As in the case of a single element, choose any point $z \in \mathcal{B}$. For each geodesic ray r emmanating from z set $m_{\Xi}(r) := \inf\{d_{\Xi}(r)\}$. Then $m_{\Xi} : \mathcal{T} \longrightarrow \mathbb{R}$ is a continuous function. Let r be a ray so that m_{Ξ} achieves it minimum. Then $m_{\Xi}(r) = d(\Xi)$. On the ray r, either

(1) $d_{\Xi} \circ r$ takes on its minimum value at a point t_1 and therefore for $x_1 = r(t_1)$, we have $d_{\Xi}(x_1) = d(\Xi)$, or
(2) the function $t \mapsto d_{\Xi}(r(t))$ decreases with limit $d(\Xi)$.

In case (2), for every $g \in \Xi$, we have $d_{\Xi} \geq d_g$ on the ray r. Since d_g is a convex function, it must decrease along the ray r. The same argument as in the preceeding paragraph applies and therefore each d_g must achieve its minimum on r at some point t_g and be constant for $t > t_g$. We can find an open compact subgroup J_g so that for all $h \in gJ_g$, $d_h \circ r$ is constant on some interval $[t_g, b]$ of positive length. We conclude that for every $g \in \Xi$, there is an open neighborhood $U_g := gJ_g \cap \Xi$ in Ξ about the point g so that for every $h \in U_g$, $d_h \circ r$ is constant on $[t_g, \infty)$. The collection $\{U_g\}$ is an open cover of the compact set Ξ; hence there is a finite subcover. This implies there is a t_2 so that for every $h \in \Xi$, the function $d_h \circ r$ is constant on $[t_2, \infty)$. We conclude that $d_\Xi(r(t_2)) = d(\Xi)$. $\square$

5.6. Define the *building set* $\mathcal{B}(\Xi)$ of a nonempty bounded subset $\Xi \subset \mathcal{G}$ to be

$$\mathcal{B}(\Xi) := \{ \ x \in \mathcal{G} \mid d_\Xi(x) = d(\Xi) \ \}. \tag{5.6.1}$$

When Ξ consists of a single element g, we shorten the notation $\mathcal{B}(\{g\})$ to $\mathcal{B}(g)$.

5.7. Corollary.
i) $\mathcal{B}(\Xi)$ is a convex subset of $\mathcal{B}$.
ii) Suppose $g, h \in \mathcal{G}$ commute. Then $h\mathcal{B}(g) \subset \mathcal{B}(g)$.
iii) If $x \in \mathcal{B}(g)$, and $n \in \mathbb{N}$ is a natural integer, then $dist(x, g^n x) = n \cdot d(g)$. In particular, if $d(g) > 0$, there is a unique geodesic $L(x)$ which contains all the points $g^i x$, $i \in \mathbb{Z}$.

Proof. The convexity of $\mathcal{B}(\Xi)$ is immediate. For assertion ii), observe that if $x \in \mathcal{B}(\Xi)$, then $\operatorname{dist}(hx, g(hx)) = \operatorname{dist}(hx, hgx) = \operatorname{dist}(x, gx) = d(g)$. To prove iii), note that a curve in $\mathcal{B}$ is a geodesic curve if and only if it is locally a geodesic curve. We claim that if $x \in \mathcal{B}(g)$, then the curve C obtained by joining the two geodesic segments $[x, gx]$ and $[gx, g^2x]$ is a geodesic curve. Indeed, let y be the midpoint of the segment $[x, gx]$. Obviously, gx is the midpoint of $[gx, g^2x]$. Since $\operatorname{dist}(y, gx) = \operatorname{dist}(gx, gy) = \frac{1}{2}\operatorname{dist}(y, gy)$, we deduce that gx is the midpoint, of the geodesic segment $[y, gy]$ and therefore the curve C is a geodesic curve in a neighborhood of gx; thus C is a geodesic curve. A simple induction shows that for any positive n, the curve formed from the geodesic segments $[g^{i-1}x, g^i x]$, $1 \leq i \leq n$ is a geodesic curve. Assertion iii) is then obvious. $\square$

Remark. Suppose s is a semisimple element which is contained in some split torus S. Then the centralizer $L = C_G(s)$ is a Levi subgroup. Corollary 5.7ii) says that the Levi subgroup acts on the building set $\mathcal{B}(s)$. An interesting question is whether $\mathcal{B}(s)$ is isometric with the extended building of L.

5.8. Proposition. *Suppose $g \in \mathcal{G}$ is the product of two commuting elements $g = hk$, with the property that the smallest closed subgroup K of $\mathcal{G}$ containing the powers of k is compact. Then $d(g) = d(h)$ and $\mathcal{B}(g) \cap \mathcal{B}(h) \neq \emptyset$.*

Proof. For each $x \in \mathcal{B}(g)$, we have

$$\operatorname{dist}(kx, gkx) = \operatorname{dist}(kx, kgx) = \operatorname{dist}(x, gx)$$

hence $k\mathcal{B}(g) \subset \mathcal{B}(g)$ and therefore $\mathcal{B}(g)$ is K invariant. Since $\mathcal{B}(g)$ is also a convex set, by the Bruhat-Tits fixed point theorem there is a point $y \in \mathcal{B}(g)$ so that $ky = y$. Then

$$d(g) = \operatorname{dist}(y, gy) = \operatorname{dist}(ky, hky) = \operatorname{dist}(ky, khy) = \operatorname{dist}(y, hy) \leq d(h). \tag{5.8.1}$$

Since, $h = gk$ is obviously a product of commuting elements, interchanging the roles of g and h, we see $d(g) = d(h)$. It then follows from (5.8.1) that $y \in \mathcal{B}(g) \cap \mathcal{B}(h)$. $\square$

5.9. Corollary. *Suppose $g = su$ is the Jordan decomposition of an element $g \in \mathcal{G}$, then*

i) $d(g) = d(s)$.

ii) The building sets $\mathcal{B}(g)$ and $\mathcal{B}(s)$ are u invariant, then the u-fixed point sets $\mathcal{B}(g)^u$ and $\mathcal{B}(s)^u$ are nonempty, and

$$\mathcal{B}(s)^u \subset \mathcal{B}(g) \quad \textit{and} \quad \mathcal{B}(g)^u \subset \mathcal{B}(s).$$

Proof. Part i) is obvious from the Proposition. The assertions in part ii) that $\mathcal{B}(g)$ and $\mathcal{B}(s)$ are u invariant is Corollary 5.7ii). That $\mathcal{B}(g)^u$ and $\mathcal{B}(s)^u$ is the nonempty follows from the Bruhat–Tits fixed point theorem. To prove the last assertion of part ii), suppose $x \in \mathcal{B}(s)^u$. Then $ux = x$ and so $sx = sux = gx$ and thus, $d(g) = d(s) = \mathrm{dist}(x, sx) = \mathrm{dist}(x, gx)$, and so $x \in \mathcal{B}(g)$. Similarly, $\mathcal{B}(g)^u \subset \mathcal{B}(s)$. □

5.10. An interesting question arising from the Corollary is whether or not $\mathcal{B}(g)^u = \mathcal{B}(g)$.

The function $g \mapsto d(g)$ is clearly a class function on $\mathcal{G}$. Corollary 5.9 says $d(g)$ depends only on the semisimple part of g. In particular, d is completely determined by its restriction to the maximal tori of $\mathcal{G}$. If T is a maximal k-torus, then Proposition 5.8 says, the function d is constant on cosets of the maximal compact subgroup $T(k)_c$ in $T(k)$.

6. A descent property

6.1. In this section we prove a descent property in $\mathcal{G}$ which is an analogue of a fundamental descent property in the Lie algebra and its dual. Recall that in [MP1,MP2] it was shown that if $\Xi = X + \mathfrak{g}^*_{x,-r_{i-1}} \in \mathfrak{g}^*_{-r_i}/\mathfrak{g}^*_{-r_{i-1}}$ is a coset which contains a nilpotent element, then $\Xi \subset \mathfrak{g}^*_{y,-s}$ for some $s < r_i$. In [MP3], this result was reinterpreted in terms of the concave depth function d^{Ξ} (notation [MP3]). In the language of depth functions, a nonzero coset Ξ contains a nilpotent element if and only if d^{Ξ} does not attain it maximum at the point x. We show here an analogue of this statement with the role of a concave depth function replaced by a convex displacement function.

Given $x \in \mathcal{B}$, let F be the smallest dimensional facet containing the point x.

6.2. Lemma. *For any $x \in \mathcal{B}$, there exists a neighborhood of the point x fixed by the subgroup $\mathcal{G}_{F,+}$.*

Proof. The set $\mathrm{Star}(F)$ is the union of all chambers which contain x and so clearly contains a neighorhood of x. As mentioned in section 3, the subspace $\mathrm{Star}(F)$ is the topological suspension of the spherical building $\mathcal{T}$ of $\mathcal{G}_F/\mathcal{G}_{F,+}$ by the facet F. The action of $\mathcal{G}_F$ on $\mathrm{Star}(F)$ is the suspension of the action of $\mathcal{G}_F$ on $\mathcal{T}$ and so $\mathrm{Star}(F)$ is fixed by the subgroup $\mathcal{G}_{F,+}$. □

6.3. Proposition. *For any $x \in \mathcal{B}$ and any $h \in g\mathcal{G}_{x,+}$, there exists a neighborhood U of x so that the two functions d_h and $d_{h\mathcal{G}_{x,+}}$ are equal on U.*

Proof. Let U be a neighborhood of x so that for every $k \in \mathcal{G}_{F,+}$ and $y \in U$, we have $y = ky$. Then $d_{hk}(y) = \operatorname{dist}(y, hky) = \operatorname{dist}(y, hy) = d_h(y)$. Hence $d_{h\mathcal{G}_{F,+}}(y) := \sup_{k \in \mathcal{G}_{F,+}} d_{hk}(y) = d_h(y)$. □

For $s > 0$, define a coset $\Xi = g\mathcal{G}_{x,+}$ of $\mathcal{G}_{x,+}$ in $\mathcal{G}_{x,s}$ to be *nondegenerate* if the depth function d_Ξ takes on its minimal value at the point x and *degenerate* if not.

6.4. Theorem. *Suppose $s > 0$ and $\Xi = gG_{x,+} \subset G_{x,s}$.*

i) If Ξ is a degenerate coset, then there exists a point y and $s' < s$ so that $\Xi \subset G_{y,s'}$.

ii) If Ξ is a nondegenerate coset, then for any nonempty subset $Y \subset \Xi$, the function d_Y achieves its minimum at the point x.

Proof. The proof of i) is obvious. If d_Ξ does not take on its minimum at the point x, there is a point y so that $s' = d_\Xi(y) < d_\Xi(x)$. But, $d_\Xi(y) = s'$ means $d_h(y) \le s'$ for all $h \in \Xi$, i.e. $\Xi \subset \mathcal{G}_{y,s}$. To prove ii), select an element $h \in Y \subset \Xi$. We have $d_h \le d_Y \le d_\Xi$. Proposition 6.3 says that $d_h = d_\Xi$ in a neighborhood of the point x; hence $d_Y = d_\Xi$ in a neighborhood of x. If $d(Y) < d(\Xi)$, let y be a point where $d_Y(y) < d(\Xi)$. Consider the geodesic segment $[y, x]$. Since d_Y is convex and has values $d_Y(y) < d(\Xi) = d_Y(x)$, it must be the case that d_Y takes on values less than $d_Y(x)$ in any neighborhood of the point x. But, this clearly contradicts the two facts that $d_Y = d_\Xi$ in a neighborhood of x and Ξ is nondegenerate. Thus, $d(Y) = d(\Xi)$, and so Y is nondegenerate. □

7. An example

7.1. We illustrate the displacement function as a tool in a proof of the Howe Conjecture for the group $G = SL(2,k)$, when $char(k) = 0$. We use the notation of [MP1] to define the filtration subgroups $G_{x,r}$ $(r \ge 0)$ attached to a point x in the building $\mathcal{B}(G)$. Denote by D the diagonal torus in G, and by A the apartment of D. Let ϖ be a prime element in $\mathfrak{o}_k$ and take $\delta \in GL(2,k)$ to be the element

$$\delta = \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let x_0 and x_1 be points in A so that $G_{x_0,0} = SL(2,\mathfrak{o}_k)$ and $G_{x_1,0} = Ad(\delta)G_{x_0,0}$. If we denote by x_h the midpoint of the segment $[x_0, x_1]$, then $G_{x_h,0}$ is the Iwahori subgroup $G_{x_0,0} \cap G_{x_1,0}$. Given a compact subset $\omega \subset G$, set $\omega^G := \{Ad(g)x \,|\, x \in \omega\,,\, g \in G\}$ and let $J(\omega)$ denote the space of G-invariant distributions supported on the closure of ω^G. Fix a positive integer r; then $G_{x_0,r}$ is the r-th congruence subgroup of $SL(2,\mathfrak{o}_k)$, $G_{x_1,r} = Ad(\delta)G_{x_0,r}$ and $G_{x_h,r} = G_{x_0,r} \cap G_{x_1,r}$. Let $\mathcal{S}(r)$ be the vector space of functions on G which are compactly supported and right $G_{x_h,r}$-invariant, i.e. $\mathcal{S}(r) = C_o^\infty(G/G_{x_h,r})$.

7.2. Group Howe Conjecture for SL(2). *Suppose $char(k) = 0$. Given a compact subset $\omega \subset SL(2,k)$ and a positive integer r, the restriction of the G-invariant distributions $J(\omega)$ to $\mathcal{S}(r)$ induces a finite dimensional space of linear functionals of $\mathcal{S}(r)$.*

Proof. As in [Ho], we view evaluation of an element $T \in J(\omega)$ on an element $f \in \mathcal{S}(r)$ as a pairing $\langle T, f\rangle := T(f)$ on $J(\omega) \times \mathcal{S}(r)$. In the dual spaces of $\mathcal{S}(r)$ and $J(\omega)$, denote by $J(\omega)|_{\mathcal{S}(r)}$ and $\mathcal{S}(r)|_{J(\omega)}$, the subspaces induced by $J(\omega)$ and $\mathcal{S}(r)$ respectively. Our plan, following both [Ho] and [HC], is to show the characteristic functions of cosets $gG_{x_h,r}$, $g \in G$ span a finite dimensional space of linear functionals on $J(\omega)$.

Set $Y = \{x_0, x_1, x_h\}$. The group $G_{x_h,r}$ is a subgroup of both $G_{x_0,r}$ and $G_{x_1,r}$; and so $C_o^\infty(G/G_{x_0,r})$ and $C_o^\infty(G/G_{x_1,r})$ are subspaces of $\mathcal{S}(r)$. For $g \in G$ and $y \in Y$, set $1_{g,y}$ to be the characteristic function of the coset $gG_{y,r}$. These functions obviously span $\mathcal{S}(r)$. We shall use the displacement function to define finite dimensional subspaces W_y in each $C_o^\infty(G/G_{y,r})$, $y \in Y$, and then give a reduction argument modeled on the one in [Ho] for Lie algebras to say that the linear functional of $J(\omega)$ induced by any $1_{g,y}$ is contained in the span of the ${W_y}'$s.

Recall the notation from section 5 that if $h \in G$, then $d(h)$ is the minimum value of the displacement function d_h. Compactness of ω means there is a $M = M(\omega) > 0$ such that $d(h) \le M$ for all $h \in \omega$. Note also for any $N > 0$, there are only finitely many cosets $gG_{y,r}$, $y \in Y$ such that $d_{gG_{y,r}}(y) \le N$.

We use the notion of degenerate and nondegenerate cosets described in section 6 to define two subspaces of $\mathcal{S}(r)$. Let $\mathcal{D}(r)$ and $\mathcal{N}(r)$ be the subspaces spanned by elements $1_{g,y}$ so that $gG_{y,0^+}$ is respectively degenerate and nondegenerate. Then, $\mathcal{S}(r) = \mathcal{D}(r) + \mathcal{N}(r)$. For convenience, call a coset $gG_{y,r}$ degenerate or nondegenerate depending on the corresponding behavior of $gG_{y,0^+}$. The crucial property of nondegenerate cosets which we use is that if $gG_{y,r}$ is nondegenerate then $d(h) = d(gG_{y,r}) = d_{gG_{y,r}}(y)$ for every $h \in gG_{y,r}$.

We claim $\mathcal{N}(r)|_{J(\omega)}$ is finite dimensional. This assertion follows from our observations that there exists an $M > 0$ so that $d(h) \le M$ for all $h \in \omega$ and that there are only finitely many cosets $gG_{y,r}$, whence only finitely many nondegenerate cosets, with $d(gG_{y,r}) \le M$, and the fact that if $gG_{y,r}$ is a nondegenerate with $d(gG_{y,r}) > M$, we have $gG_{y,r} \cap \omega^G = \emptyset$, and so $1_{g,y}|_{J(\omega)} \equiv 0$.

For $N > 0$, let J_N^* be the subspace of the dual space $J(\omega)^*$ spanned by $\mathcal{N}(r)|_{J(\omega)}$ and the functions $1_{g,y}|_{J(\omega)}$, where $y \in Y$ and $gG_{y,r}$ satisfies $d_{gG_{y,r}}(y) \le N$. The results of the previous paragraph implies J_N^* is finite dimensional. We shall show that if $gG_{y,r}$ is a degenerate coset, and $R = d_{gG_{y,r}}(y)$ is sufficiently large, then the linear functional of $J(\omega)$ induced by $1_{g,y}$ in fact lies in the subspace J_{R-1}^*. A proof of this assertion establishes 7.2. To accomplish our goal we first prove a result on degenerate cosets. For convenience, normalize the distance function on $\mathcal{B}$ so that $\mathrm{dist}(x_0, x_1) = 1$. If S is any split torus in G, denote by S_c the maximal compact subgroup of S.

7.3. Lemma. *Suppose $y \in Y$ and $gG_{y,0^+}$ is a degenerate coset, then there exists an apartment $A(S)$ containing y, a unipotent element u so that $u - 1 \in \mathfrak{sl}(2,k)$ is a eigenvector* *for $Ad(S)$, and an element $a \in S_c$ so that $gG_{y,0^+} = auG_{y,0^+}$.*

Proof. Call a point $z \in \mathcal{B}$ a vertex point if $G_{z,0^+}$ is a maximal compact subgroup of G. It is elementary to see, e.g. via a graph, that if $gG_{y,0^+}$ is a degenerate coset, then $gy \neq y$ and furthermore

i) $d_{gG_{y,0^+}}$ decreases in a neighborhood of the point y in the geodesic $[y, gy]$.
ii) If $y \in \{x_0, x_1\}$, then $\operatorname{dist}(y, gy)$ is an even integer; while if $y = x_h$, then $\operatorname{dist}(y, gy)$ is an odd integer. In any case, the midpoint z of $[y, gy]$ is a vertex.

Let $A(S)$ be any apartment which passes through the points y and z and does not meet the segment $(z, gy]$. Denote by z_a, the vertex in $(y, z]$ closest to y and z_b the vertex in $A(S)$ on the opposite side of y at distance $\operatorname{dist}(y, z_a)$. It is again elementary that there exists a unipotent element, so that $u - 1 \in \mathfrak{sl}(2, k)$ is a eigenvector for $Ad(S)$, so that the half line of $A(S)$ containing z but not y is fixed, and such that $uz_a = gz_a$, $uy = gy$, and $uz_b = gz_b$. It follows that $g^{-1}u \in G_{z_a,0} \cap G_{y,0} \cap G_{z_b,0} = S_c G_{y,0^+}$, which immediately implies the lemma. $\square$

We are now ready to complete the proof of 7.2. Suppose $y \in \{x_0, x_1\}$ and $gG_{y,r}$ is degenerate. Set $d_{gG_{y,r}}(y) = R$. As our argument for x_0 and x_1 are similar, we'll assume $y = x_0$. Apply Lemma 7.3 to find an apartment $A(S)$ containing y and $gG_{y,r} = aukG_{y,r}$, where $k \in G_{y,0^+}$. Since we are only interested in the linear functional of $J(\omega)$ induced by $1_{g,y}$, we can replace $gG_{y,r}$ by a conjugate $g'G_{y,r} = Ad(k')gG_{y,r}$ ($k' \in G_{y,0}$). By replacing $gG_{y,r}$ by a suitable conjugate $g'G_{y,r}$, we may assume the torus we obtain from Lemma 7.3 is the diagonal torus D and u is an upper triangular unipotent element. The coset $g'G_{y,r}$ is a disjoint union $\coprod g'm_i G_{x_h,r}$, where m_i runs over $G_{y,r}/G_{x_h,r}$ and $d_{g'm_iG_{x_h,r}}(x_h) = R - 1$. Hence, $1_{g',y} = \sum 1_{g'm_i,x_h} \in J^*_{R-1}$.

The last case we need to consider is a degenerate coset of the form $gG_{x_h,r}$. Set $R = d_{gG_{x_h,0^+}}(x_h)$. Arguing as in the previous paragraph, after a possible conjugation, we can assume the torus provided by Lemma 7.3 is the diagonal torus D, and $gG_{x_h,r} = aukG_{x_h,r}$, with $k \in G_{x_h,0^+}$. Our argument is essentially the same for u an upper or lower triangular unipotent element, so we'll assume u is upper triangular. In this situation, $d_{gG_{x_h,r}}(x_1) = R - 1$. An extra complication, because $G_{x_h,r}$ is a proper subgroup of $G_{x_1,r}$, is that the coset $gG_{x_h,r}$ is not a disjoint union of $G_{x_1,r}$ cosets. In fact, we can choose representatives of the cosets in $G_{x_1,r}/G_{x_h,r}$ to be upper triangular unipotent elements. However, provided R (which equals $d_u(x_1)$) is large enough, we can conjugate $gG_{x_h,r} = aukG_{x_h,r}$ by a small open compact subgroup H of D_c (which means we effectively only conjugate the unipotent element u by H) so that $Ad(H)(gG_{x_h,r}) = a(Ad(H)u)kG_{x_h,r} = aukG_{x_1,r}$. It is this step that requires $char(k) = 0$, so that $(k^\times)^2$ contains an open neighborhood of 1. We conclude there exists a $L > 0$ so that if $R > L$, then $1_{g,y} \in J^*_{R-1}$. Finally, if $M = M(\omega)$ is the bound we defined during our discussion of nondegenerate cosets, then the space of linear functionals of $J(\omega)$ induced by $\mathcal{S}(r)$ equals the finite dimensional space $J^*_{\max(L,M)}$. $\square$

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