INVOLUTIONS, CLASSICAL GROUPS AND BUILDINGS

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ABSTRACT. In [CLT], Curtis-Lehrer-Tits construct a variation of the Tits building. The Curtis-Lehrer-Tits building $\mathcal{L}(\mathsf{G},k)$ of a connected reductive k-group G has the important feature that it is a functor from the category of reductive groups defined over a field k and monomorphisms to the category of topological spaces and inclusions. An important consequence derived by Curtis-Lehrer-Tits from the functorial nature of the Curtis-Lehrer-Tits building $\mathcal{L}(\mathsf{G},k)$ is that if s is a semisimple element of the group $\mathsf{G}(k)$ of k-rational points, and G' is the connected component group of the centralizer of s, then the fixed point set $\mathcal{L}(\mathsf{G},k)^s$ of s in $\mathcal{L}(\mathsf{G},k)$ is the Curtis-Lehrer-Tits building $\mathcal{L}(\mathsf{G}',k)$. We generalize this result to arbitrary involutions of $\mathrm{Aut}_k(\mathsf{G})$, and we also prove an analogue in the context of affine buildings.

1. INTRODUCTION

1.1. Let k be a field and G a connected reductive group defined over k. The Tits or spherical combinatorial building $\Delta(\mathsf{G},k)$ of G with respect to k is the simplicial complex whose simplices are the proper parabolic k-subgroups of G, reverse ordered by inclusions. A parabolic k-subgroup P is an r-simplex if and only if there are $(r+1)$ distinct $\mathsf{P}_0, \mathsf{P}_1, \ldots, \mathsf{P}_r$ maximal parabolic k-subgroups of G so that $\mathsf{P} = \mathsf{P}_0 \cap \mathsf{P}_1 \cap \cdots \cap \mathsf{P}_r$. The group $\mathcal{G} := \mathsf{G}(k)$ of k-rational points of G operates simplicially on $\Delta(\mathsf{G},k)$. In [CLT], Curtis-Lehrer-Tits construct a variation of the Tits building. The Curtis-Lehrer-Tits building $\mathcal{L}(\mathsf{G},k)$ of G has the important feature that it is a functor from the category of connected reductive groups defined over k and monomorphisms to the category of topological spaces and inclusions. That is, given a monomorphism $f : \mathsf{G} \to \mathsf{H}$ of reductive groups defined over k, Curtis-Lehrer-Tits naturally associate an embedding of topological spaces $\mathcal{L}(f) : \mathcal{L}(\mathsf{G},k) \to \mathcal{L}(\mathsf{H},k)$. An important consequence derived by Curtis-Lehrer-Tits from the functorial nature of the Curtis-Lehrer-Tits building $\mathcal{L}(\mathsf{G},k)$ is the following: Suppose s is a semisimple element of $\mathcal{G}$. Let $\mathsf{G}' := (\mathsf{G}^s)^\circ$ denote the connected centralizer of s in G. Then, the fixed point set $\mathcal{L}(\mathsf{G},k)^s$ of s in $\mathcal{L}(\mathsf{G},k)$ is the Curtis-Lehrer-Tits building $\mathcal{L}(\mathsf{G}',k)$. One of our main goals (Theorem 3.3.1) is to establish, when $char(k) \neq 2$, a generalization of the Curtis-Lehrer-Tits result to arbitrary involutions in $\mathrm{Aut}_k(\mathsf{G})$.

1.2. Suppose k is a nonarchimedean local field, and G is a connected semisimple k-group. Suppose further that τ is a k-automorphism of G so that the k-group $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple. Let $\mathcal{B}(\mathsf{G}, k)$ and $\mathcal{B}(\mathsf{G}', k)$ denote the Bruhat–Tits affine buildings of G and G' respectively. It is natural to ask what is the relation between $\mathcal{B}(\mathsf{G}, k)^\tau$ and $\mathcal{B}(\mathsf{G}', k)$. Under the assumption that the residual characteristic of k is odd, G is a special linear group G and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution defining a classical group, we show (Theorem 6.7.3) $\mathcal{B}(\mathsf{G}, k)^\tau$ can be identified with $\mathcal{B}(\mathsf{G}', k)$. This type of result is related to the question of when Bruhat–Tits buildings are functorial [L].

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2. Preliminaries on the Curtis-Lehrer-Tits building.

2.1. Let G be a connected reductive group defined over a field k and let $\mathcal{G}$ denote the group of k-rational points $\mathsf{G}(k)$. We review the construction of the Curtis-Lehrer-Tits (spherical) building $\mathcal{L}(\mathsf{G}, k)$ of G in [CLT]. Let S be a maximal k-split torus of G and let $\Phi(\mathsf{S}, \mathsf{G})$ be the k-roots of G with respect to S. Denote the k-cocharacters $\mathrm{Hom}_k(\mathrm{GL}(1), \mathsf{S})$ of S as $X_*(\mathsf{S}, k)$. The space $\mathcal{L}(\mathsf{S}, k)$ is defined as the sphere whose points represent rays in the real vector space $X_*(\mathsf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$. To any point $b \in \mathcal{L}(\mathsf{S}, k)$, i.e. a ray in $X_*(\mathsf{S}, k) \otimes_{\mathbb{Z}} \mathbb{R}$, we associate the parabolic k-subgroup $\mathsf{P}(b)$ defined as the largest closed subgroup of G containing S and whose Lie algebra contains the roots $\psi \in \Phi(\mathsf{S}, \mathsf{G})$ whose inner product with b is non-negative.

2.2. Denote as $\mathcal{L}_1$ the disjoint union of the spheres $\mathcal{L}(\mathsf{S}, k)$ as S runs over all the maximal k-split tori of G. The group $\mathrm{Aut}_k(\mathsf{G})$ of k-automorphisms of G obviously acts on $\mathcal{L}_1$. In particular, $\mathcal{G}$ acts on $\mathcal{L}_1$ by inner automorphisms. We define an equivalence on points in $\mathcal{L}_1$ as follows. Given $b \in \mathcal{L}(\mathsf{S}, k)$ and $b' \in \mathcal{L}(\mathsf{S}', k)$, we define $b \sim b'$ if $\mathsf{P}(b) = \mathsf{P}(b')$ and there is a $g \in \mathsf{P}(b)$ conjugating S to S' so that $b' = Ad(g)(b)$. The *Curtis-Lehrer-Tits (spherical) building* $\mathcal{L}(\mathsf{G}, k)$ of G with respect to k is defined as the quotient of $\mathcal{L}_1$ by the equivalence relation $\sim$. It is immediate from (2.1) that $b \in \mathcal{L}(\mathsf{G}, k)$ determines unambiguously a parabolic k-subgroup $\mathsf{P}(b)$ of G. The building $\mathcal{L}(\mathsf{G}, k)$ inherits an action by $\mathrm{Aut}_k(\mathsf{G})$; in particular an action of $\mathcal{G}$.

For our purposes, the field k is fixed. Therefore, in order to simplify notation, we shall often use $X_*(\mathsf{S})$, $\mathcal{L}(\mathsf{S})$, and $\mathcal{L}(\mathsf{G})$ to denote $X_*(\mathsf{S}, k)$, $\mathcal{L}(\mathsf{S}, k)$, and $\mathcal{L}(\mathsf{G}, k)$ respectively.

2.2.1. Lemma. (Lemma 2.2 of [CLT])

i) The stabilizer of a point $b \in \mathcal{L}(\mathsf{G})$ in $\mathcal{G}$ is $\mathsf{P}(b)(k)$.

ii) Let S be a maximal k-split torus in G. The canonical projection $\mathcal{L}_1 \to \mathcal{L}(\mathsf{G})$ restricts to an injection of $\mathcal{L}(\mathsf{S})$ into $\mathcal{L}(\mathsf{G})$.

The image of $\mathcal{L}(\mathsf{S})$ in $\mathcal{L}(\mathsf{G})$ is called the (*spherical*) *apartment* associated with S in $\mathcal{L}(\mathsf{G})$. For convenience, we identify $\mathcal{L}(\mathsf{S})$ with its corresponding apartment. A point b lies in an apartment $\mathcal{L}(\mathsf{S})$ precisely when $\mathsf{S} \subset \mathsf{P}(b)$.

2.2.1. Lemma. (statements (2.3) and (2.4) of [CLT])

iii) Any two points of $\mathcal{L}(\mathsf{G})$ *belong to an apartment.*

iv) If S *and* S' *are two maximal* k*-split tori, there is an element* $g \in \mathcal{G}$ *conjugating* S *to* S' *(hence* g *takes* $\mathcal{L}(\mathsf{S})$ *to* $\mathcal{L}(\mathsf{S}')$*), which fixes all points in* $\mathcal{L}(\mathsf{S}) \cap \mathcal{L}(\mathsf{S}')$.

2.3. Let S' be any k-split torus and let S be a maximal k-split torus containing S'. The canonical injection $X_*(\mathsf{S}') \to X_*(\mathsf{S})$ induces an injection $\mathcal{L}(\mathsf{S}') \to \mathcal{L}(\mathsf{S})$ and hence an injection of $\mathcal{L}(\mathsf{S}')$ into $\mathcal{L}(\mathsf{G})$. As a map into $\mathcal{L}(\mathsf{G})$, this injection is independent of the torus S. In particular, any point $b \in \mathcal{L}(\mathsf{S})$ determines a parabolic k-subgroup $\mathsf{P}(b) \subset \mathsf{G}$.

2.3.1. Lemma. (Lemma 1.2 of [CLT]) *Let* S *be a* k*-split torus in* G.

i) For $b \in \mathcal{L}(\mathsf{S})$*, we have* $\mathsf{P}(b) \supset \mathsf{C}_{\mathsf{G}}(\mathsf{S})$.

ii) For $b \in \mathcal{L}(\mathsf{S})$*, let* S' *be the intersection of* S *with the radical of* $\mathsf{P}(b)$*. Then* $b \in \mathcal{L}(\mathsf{S}')$.

2.4. Suppose G' is a connected reductive k-group and $f : \mathsf{G}' \to \mathsf{G}$ is a k-monomorphism. Let $\mathcal{L}_1'$ be the disjoint union of all $\mathcal{L}(\mathsf{S}')$ with S' a maximal k-split torus of G'. The map $f : \mathsf{G}' \to \mathsf{G}$ induces a mapping $\mathcal{L}_1' \to \mathcal{L}(\mathsf{G})$. It is shown in [CLT:§4] that two points $b, b' \in \mathcal{L}_1'$ are equivalent under the equivalence relation $\sim_{\mathsf{G}'}$ described by G' if and only if they have the same image in $\mathcal{L}(\mathsf{G})$. Thus, the map $f : \mathsf{G}' \to \mathsf{G}$ induces an injective map $\mathcal{L}(f) : \mathcal{L}(\mathsf{G}') \to \mathcal{L}(\mathsf{G})$; or equivalently the Curtis-Lehrer-Tits building is a functor from the category of connected reductive k-groups with k-monomorphisms into the category of sets with injections.

2.5. As mentioned in the introduction, an important consequence of the functorial nature of the Curtis-Lehrer-Tits building is the following proposition.

2.5.1. Proposition. (Proposition 5.1 of [CLT]) *Let* s *be a* k*-rational semisimple element of a connected reductive* k*-group* G *and denote the connected component group of the centralizer of* s *as* $\mathsf{G}' = \mathsf{C}_{\mathsf{G}}(s)^\circ$*. Then,* $\mathcal{L}(\mathsf{G}', k)$ *is the fixed point set* $\mathcal{L}(\mathsf{G}, k)^s$ *of* s *acting on* $\mathcal{L}(\mathsf{G}, k)$.

3. Involutions and spherical buildings

3.1. Suppose $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution, i.e. of order two. From now on, we assume $\mathrm{char}(k) \neq 2$. Under this assumption, the induced action of τ on $\mathrm{Lie}(\mathsf{G})$ is semisimple, and therefore $\mathrm{Lie}(\mathsf{G}^\tau)$ and G^τ are reductive. For notational ease, we let G' denote the connected component group $(\mathsf{G}^\tau)^\circ$. Our main goal in this section is to prove

$$\mathcal{L}(\mathsf{G}, k)^\tau = \mathcal{L}(\mathsf{G}', k).$$

3.2. We begin with a preliminary lemma.

3.2.1. Lemma.

i) If S is a maximal k-split torus in a linear algebraic k-group Q, then the intersection $\mathsf{S}' = \mathsf{S} \cap R(\mathsf{Q})$ of S with the radical $R(\mathsf{Q})$ of Q is a maximal k-split torus in $R(\mathsf{Q})$.

ii) In the situtation of i), any two maximal k-split tori in $R(\mathsf{Q})$ are conjugate by an element $u \in R_u(\mathsf{Q})(k)$, where $R_u(\mathsf{Q})$ is the unipotent radical of Q.

iii) Suppose $char(k) \neq 2$. If U is a unipotent k-group then if $u \in \mathsf{U}(k)$, there is a unique $v \in \mathsf{U}(k)$ such that $u = v^2$.

3.2.2. *Remark.* A trivial modification of the proof of Lemma 3.2.1 iii), shows under the assumption ℓ is a prime and $char(k) \neq \ell$, if $u \in \mathsf{U}(k)$, then there is a unique $v \in \mathsf{U}(k)$, so that $u = v^\ell$.

Proof. To prove (i), let T' be a maximal split k-torus in $R(\mathsf{Q})$ and $\mathsf{T}' \subset \mathsf{T}$ a maximal split k-torus in Q. There exists $g \in \mathsf{Q}(k)$ such that $g\mathsf{T}g^{-1} = \mathsf{S}$. Thus, $\mathsf{S}' = \mathsf{S} \cap R(\mathsf{Q}) = g\mathsf{T}g^{-1} \cap R(\mathsf{Q}) = g(\mathsf{T} \cap R(\mathsf{Q}))g^{-1} = g\mathsf{T}'g^{-1}$ must be a maximal split k-torus in $R(\mathsf{Q})$. Statement (ii) is [B1:11.23]. To prove statement (iii), let $\mathsf{U} = \mathsf{U}_0 \supset \mathsf{U}_1 \supset \cdots \supset \mathsf{U}_{r-1} \supset \mathsf{U}_r = \{0\}$ be the descending central series of U, i.e. for $i \geq 1$, U_i is defined as the group $\mathsf{U}_i := [\mathsf{U}, \mathsf{U}_{i-1}]$ generated by the commutators of elements in U with those in U_{i-1}. We note that $\mathsf{U}(k)/\mathsf{U}_i(k) = (\mathsf{U}/\mathsf{U}_i)(k)$. Define $\mathcal{U} := \mathsf{U}(k)$ and $\mathcal{U}_i := \mathsf{U}_i(k)$. For $1 \leq i \leq r$, we claim there is a unique sequence of elements $\overline{v}_i \in \mathcal{U}/\mathcal{U}_i$, so that $u\mathcal{U}_i = \overline{v}_i^2$ in $\mathcal{U}/\mathcal{U}_i$. When $i = 1$, this assertion reduces to the statement that $\mathcal{U}/\mathcal{U}_1$ is a vector space over k. In particular, the condition on the characteristic of k is now clear. Suppose now by way of induction that the assertion is true for i, i.e. there is a unique $\overline{v}_i \in \mathcal{U}/\mathcal{U}_i$ so that $u\mathcal{U}_i = \overline{v}_i^2$ in $\mathcal{U}/\mathcal{U}_i$. Pick a representative v_i of $\overline{v}_i$. Then $v_i^{-2}u \in \mathcal{U}_i$. But again, $\mathcal{U}_i/\mathcal{U}_{i+1}$ is a vector space over k so there is $y_i \in \mathcal{U}_i$ which is unique module $\mathcal{U}_{i+1}$ so that $u = v_i^2 y_i^2$ modulo $\mathcal{U}_{i+1}$. But $v_i^2 y_i^2 = v_i y_i y_i^{-1} v_i y_i y_i = v_i y_i (y_i^{-1} v_i y_i v_i^{-1}) v_i y_i = (v_i y_i)^2$ modulo $\mathcal{U}_{i+1}$. Thus $\overline{v}_{i+1} = v_i y_i$ modulo $\mathcal{U}_{i+1}$ has the property $u = \overline{v}_{i+1}^2$ modulo $\mathcal{U}_{i+1}$. Uniqueness of $\overline{v}_{i+1}$ follows from the combination of uniqueness of $\overline{v}_i$ and y_i in $\mathcal{U}/\mathcal{U}_i$ and $\mathcal{U}_i/\mathcal{U}_{i+1}$ respectively and that $\mathcal{U}_{i+1} \subset [\mathcal{U}, \mathcal{U}_i]$. □

3.3. We now prove this section's main result.

3.3.1. Theorem. *Suppose $char(k) \neq 2$, G is a connected reductive k-group, $\tau \in Aut_k(\mathsf{G})$ is an involution and $\mathsf{G}' = (\mathsf{G}^\tau)^\circ$. Then*

$$\mathcal{L}(\mathsf{G}, k)^\tau = \mathcal{L}(\mathsf{G}', k).$$

Proof. Let S be a maximal k-split torus. Suppose $b \in \mathcal{L}(\mathsf{S}) \subset \mathcal{L}(\mathsf{G})$. Let $\mathsf{P}(b)(k)$ be the parabolic subgroup which fixes b. By Lemma 3.2.1 (i), the split k-torus $\mathsf{S}' = \mathsf{S} \cap R(\mathsf{P}(b))$ is a maximal split k-torus in $R(\mathsf{P}(b))$, and by Lemma 2.3.1 (ii), we have $b \in \mathcal{L}(\mathsf{S}')$. Of course, $\tau(\mathsf{S}')$ is a maximal split k-torus in $R(\tau(\mathsf{P}(b)))$ and $\tau(b) \in \mathcal{L}(\tau(\mathsf{S}'))$. If b is fixed by τ, then $\mathsf{P}(b)$ is τ invariant and therefore S' and $\tau(\mathsf{S}')$ are maximal k-split tori in $R(\mathsf{P}(b))$. By Lemma 3.2.1 (ii), there is a $u \in R_u(\mathsf{P}(b))(k)$ so that $\tau(\mathsf{S}') = u\mathsf{S}'u^{-1}$. If we apply τ to this last equality, we obtain $\mathsf{S}' = \tau(u)\tau(\mathsf{S}')\tau(u^{-1}) = \tau(u)u\mathsf{S}'u^{-1}\tau(u^{-1})$. Clearly $R_u(\mathsf{P}(b))$

is τ-invariant, so $\tau(u)u \in R_u(\mathsf{P}(b))$ and $\tau(u)u$ normalizes S'. Since S' is a maximal split k-torus in $R(\mathsf{P}(b))$, we conclude $\tau(u)u = 1$. By Lemma 3.2.1 (iii), there is a unique element $v \in R_u(\mathsf{P}(b))(k)$ so that $u = v^2$. Therefore $v^2 = u = \tau(u^{-1}) = \tau(v^{-1})^2$, i.e. v and $\tau(v^{-1})$ are square roots of u in $R_u(\mathsf{P}(b))(k)$, and therefore by uniqueness of square roots, $\tau(v^{-1}) = v$. Consider the maximal split k-torus in $R(\mathsf{P}(b))$ which is $\mathsf{T} = v\mathsf{S}'v^{-1}$. Then

$$\tau(\mathsf{T}) = \tau(v)\tau(\mathsf{S}')\tau(v^{-1}) = v^{-1}u\mathsf{S}'u^{-1}v = v\mathsf{S}'v^{-1} = \mathsf{T}.$$

That is, T is a τ-stable torus. Clearly $b \in \mathcal{L}(\mathsf{T})$. So, b is a ray in $X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ which is τ invariant. This means b is a ray in $(X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R})^\tau = X_*(\mathsf{T}^\tau) \otimes_{\mathbb{Z}} \mathbb{R}$. A ray in $X_*(\mathsf{T}^\tau) \otimes_{\mathbb{Z}} \mathbb{R}$ is precisely a point in $\mathcal{L}(\mathsf{T}^\tau) \subset \mathcal{L}(\mathsf{G}', k)$. Thus, $\mathcal{L}(\mathsf{G}, k)^\tau = \mathcal{L}(\mathsf{G}', k)$. $\square$.

3.4. The next result shall be used when we discuss involutions and affine buildings. Geometrically, it implies that when k is a finite field of odd order and τ is a nontrivial involution of G, then any point $b \in \mathcal{L}(\mathsf{G}, k)^\tau$ lies in a τ-invariant apartment of $\mathcal{L}(\mathsf{G}, k)$.

3.4.1. Proposition. *Suppose k is a finite field of odd order q, G is a connected reductive k-group, $\tau \in Aut_k(\mathsf{G})$ is an involution and $\mathsf{G}' = (\mathsf{G}^\tau)^\circ$. If T is a k-split torus in G', then there exists a maximal k-split torus S in G which contains T and is τ-invariant.*

Proof. Let $\mathsf{M} = \mathsf{C}_{\mathsf{G}}(\mathsf{T})$ be a Levi k-subgroup of G. We make three trivial observations.

i) The group M is τ-invariant.
ii) Any maximal k-split torus in M must contain the torus T.
iii) The reductive k-rank of M and G are equal. In particular, a maximal split k-torus in M is also a maximal split k-torus in G. Denote the rank by r.

Let B (resp. U) be a Borel k-subgroup of M (resp. the unipotent radical of B). Then, the number of maximal split k-tori in B is equal to the order of the group $\mathsf{U}(k)$. The k-subgroup $\tau(\mathsf{B})$ is obviously also a Borel subgroup of M. Let Ω denote the set of maximal split k-tori in $\mathsf{B} \cap \tau(\mathsf{B})$. Obviously, since $\mathsf{B} \cap \tau(\mathsf{B})$ is a τ-invariant group, the set Ω is a τ-invariant set. The intersection of any two minimal parabolic k-subgroups must contain a minimal Levi k-subgroup. Thus, the intersection $\mathsf{B} \cap \tau(\mathsf{B})$ contains a minimal Levi k-subgroup and so a k-torus of dimension r, i.e. Ω is nonempty. Any two elements in Ω are conjugate in $\mathsf{B} \cap \tau(\mathsf{B})$. We conclude Ω has order a power of $q = |k|$, in particular has odd order. Thus, there must be a τ-fixed point S in Ω. The torus S is our desired torus. $\square$

4. Simple algebras, involutions and classical groups

4.1. In the previous section, under the assumption that k is a field of odd characteristic, G is connected reductive k-group, $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$, we showed $\mathcal{L}(\mathsf{G}, k)^\tau = \mathcal{L}(\mathsf{G}', k)$. In this section, we recall the relationship between the simple classical groups and simple algebras with involutions.

Let k be a field and suppose A is a finite dimensional simple algebra over k. A fundamental result of Weddeburn asserts that A is isomorphic as a k-algebra to a matrix algebra $M_n(D)$ of a finite dimensional division algebra D over k. Furthermore, if D_1 and D_2 are two finite dimensional algebras over k and $M_{n_1}(D_1)$ and $M_{n_2}(D_2)$ are isomorphic as k-algebras, then $n_1 = n_2$ and D_1 and D_2 are isomorphic k-algebras.

4.2. A k-involution of a k-algebra A is an invertible k-linear map $J : A \to A$ satisfying

$$J^2 = I \text{ (Identity) }, \qquad \text{and} \qquad J(ab) = J(b)J(a). \tag{4.2.1}$$

An element $a \in A$ is *J-symmetric* if $J(a) = a$ and *J-skew* if $J(a) = -a$. The sets of J-symmetric and J-skew elements are trivially k-linear subspaces of A. If the characteristic of k is not two, then A is the additive direct sum of these two linear subspaces.

4.3. Involutions fall into two distinct types. Suppose J is a k-involution of A. Since J is surjective, it must preserve the center C of the algebra A. Since we are assuming A is a simple algebra, the center C must be a finite dimensional field extension of k. The involution J is said to be of the *first* or *second kind* depending on whether C lies in the symmetrical elements of J. That is, J is of the first kind if the center C is elementwise fixed by J, and J is of the second kind if there are elements in the center C which are not J symmetric. When J is an involution of the second kind, the symmetric elements C^J in the center C is a finite extension of k so that $[C : C^J] = 2$.

Let C be the center of a simple finite dimensional k-algebra A and suppose S is a k-subfield of C. We call a k-involution J of A an *involution over S of A* if the J-symmetric elements in C is precisely S. We remark that C is either equal to S or a quadratic extension of S.

4.3.1. Theorem. (Theorem 11 page 154 of [A]) *Let C be the center of a simple finite dimensional k-algebra A. Suppose S is a k-subfield of C and J is an involution of A over S. Then, a k-anti-isomorphism $T : A \to A$ is a k-involution over S if and only if there exists a J-symmetric or J-skew invertible element $y \in A$, i.e. $J(y) = \pm y$, such that*

$$T(a) = y^{-1}\, J(a)\, y \quad a \in A.$$

4.3.2. Let D be a finite dimensional dvision algebra over k and let $A = M_n(D)$. Denote the identity matrix in A as I. We identify $d \in D$ with the matrix $dI \in A$. The center C of D becomes the center of A. For $1 \le i, j \le n$, let $e_{i,j}$ be the matrix whose r, s entry is $\delta_{i,r}\delta_{j,s}$ (Kronecker delta).

4.3.3. Theorem. (Theorem 12 page 156 of [A]) *Let D be a finite dimensional division algebra over k and set $A = M_n(D)$. Let C be the center of D and S a k-subfield of C. Then, A has a k-involution over S if and only if there exists a k-involution of D over S. In this case, A has a k-involution J over S such that $J(D) = D$ and $J(e_{i,j}) = e_{j,i}$.*

4.4. Suppose D is a finite dimensional dvision algebra over k. For $a \in M_n(D)$, denote by a^t the transpose of a. Also, if J is a k-anti-isomorphism of D, and $a = (a_{i,j}) \in A$ set

$J(a) := (J(a_{i,j}))$. As a consequence of Theorems 4.3.1 and 4.3.3, we conclude that any k-involution T of $A = M_n(D)$ over $S \subset C$ is has the form

$$T(a) = y^{-1}\, J(a^t)\, y \quad a \in A \tag{4.4.1}$$

where J is a k-involution of D over S and $y \in A^\times$ satisfies $J(y^t) = \pm y$.

Let T be an involution of $A = M_n(D)$. Write T as in (4.4.1). Set $\mathcal{G} := A^\times = GL_n(D)$ and define $\tau : \mathcal{G} \to \mathcal{G}$ by

$$\tau(a) := T(a^{-1}). \tag{4.4.2}$$

Set

$$\mathcal{G}^\tau := \{\ a \in \mathcal{G} \mid \tau(a) = a\ \}. \tag{4.4.3}$$

Let $V := D^n$ be the (A, D)-bimodule of size n column vectors with entries in D. We use the element y, which is either J-symmetric or J-skew, to define a form $\langle\, ,\, \rangle$ on V:

$$\langle v, w\rangle := J(w^t)\, y\, v \tag{4.4.4}$$

In particular, if $d_1, d_2 \in D$, we have

$$\langle v\, d_1, w\, d_2\rangle \;=\; J(d_1)\, \langle v, w\rangle\, d_2\,.$$

Define $\epsilon \in \{\pm 1\}$ by $J(y^t) = \epsilon y$. The form $\langle\, ,\, \rangle$ is ϵ-hermitian, i.e.

$$J\big(\langle v, w\rangle\big) \;=\; \epsilon\langle w, v\rangle\,. \tag{4.4.5}$$

In terms of the form $\langle\, ,\, \rangle$, the group $\mathcal{G}^\tau$ has the description

$$\begin{aligned} \mathcal{G}^\tau &= \{\ a \in \mathcal{G} \mid \tau(a) = a\ \} \\ &= \{\ a \in \mathcal{G} \mid \langle av, aw\rangle = \langle v, w\rangle\ \forall\, v, w \in V\}. \end{aligned} \tag{4.4.6}$$

The two groups $\mathcal{G}$ and $\mathcal{G}^\tau$ are the k-rational points of two reductive k-groups Γ and Γ^τ. Their explicit description as functors is the following: if K is an extension field of k then

$$\Gamma(K) = (A \otimes_k K)^\times \tag{4.4.7}$$

and

$$\Gamma^\tau(K) = \{\, g \in \Gamma(K) \mid \tau(g) = g\, \}. \tag{4.4.8}$$

The k-group Γ (resp. Γ^τ) is of course a general linear (resp. classical) group. Recall the reduced norm is a k-group homomorphism

$$\mathrm{N}_{D/k} : \Gamma \longrightarrow \mathsf{GL}(1).$$

Define $\mathsf{G} :=$ ker ($\mathrm{N}_{D/k}$) and $\mathsf{G}^\tau :=$ ker ($\mathrm{N}_{D/k}|_{\Gamma^\tau}$). Then,

$$\mathsf{G}(K) = \{\, g \in \Gamma(K) \mid \ \mathrm{N}_{D/k}(g) = 1 \,\} \tag{4.4.9}$$

and

$$\mathsf{G}^\tau(K) = \mathsf{G}(K) \cap \Gamma^\tau(K). \tag{4.4.10}$$

5. Involutions and affine buildings I

5.1. Our goal in this and the next section is to prove an analogue of Theorem 3.3.1 for affine buildings. Let k be a nonarchimedean local field with ring of integers $\mathfrak{o}$ and residue field $\mathfrak{f}$ of odd characteristic. Suppose G is a connected semisimple group defined over k. Set $\mathcal{G} = \mathsf{G}(k)$ and let $\mathcal{B} = \mathcal{B}(\mathsf{G}, k)$ denote the Bruhat-Tits building of G with respect to k.

We set some notation. Let r denote the k-rank of G. A *face* E of a (closed) chamber C of $\mathcal{B}$ is the closure of a $(r-1)$-facet of C. If $x \in \mathcal{B}$, denote by G_x the reductive $\mathfrak{f}$-group which is the reduction modulo $\mathfrak{p}$ (the prime ideal of $\mathfrak{f}$) of the $\mathfrak{o}$-group scheme associated to the parahoric subgroup fixing the point x. Let $\mathrm{Star}(x)$ denote the union of the (closed) chambers of $\mathcal{B}$ containing the point x. If S is a maximal k-split torus, let $A(\mathsf{S}, k)$ denote the apartment of S. Like our previous situation for the Curtis-Lehrer-Tits building, we will often drop k in our notation. Thus, we shorten $\mathcal{B}(\mathsf{G}, k)$ to $\mathcal{B}(\mathsf{G})$ and $A(\mathsf{S}, k)$ to $A(\mathsf{S})$. Suppose $A(\mathsf{S})$ is an apartment containing the point x. Then S determines a $\mathfrak{o}$-scheme whose reduction modulo $\mathfrak{p}$ is a maximal $\mathfrak{f}$-split torus S_x in G_x. Every maximal $\mathfrak{f}$-split torus in G_x is obtained as an S_x. If S and S' are two maximal k-split tori with $x \in A(\mathsf{S})$ and $x \in A(\mathsf{S}')$, then $\mathsf{S}_x = \mathsf{S}'_x$ precisely when $A(\mathsf{S}) \cap \mathrm{Star}(x) = A(\mathsf{S}') \cap \mathrm{Star}(x)$.

5.2. Let $\mathcal{L}(x)$ denote the set of geodesic rays in $\mathrm{Star}(x)$ which begin at the point x. This set can be identified with the Curtis-Lehrer-Tits building of G_x as follows. Given $l \in \mathcal{L}(x)$, let S be a maximal k-torus with $l \subset A(\mathsf{S})$. The ray l determines a unique ray in $\mathrm{Hom}_k(\mathrm{GL}(1), \mathsf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}_{\mathfrak{f}}(\mathrm{GL}(1), \mathsf{S}_x) \otimes_{\mathbb{Z}} \mathbb{R}$, i.e. a point $\theta_{\mathsf{S}_x}(l)$ in $\mathcal{L}(\mathsf{S}_x, \mathfrak{f}) \subset \mathcal{L}(\mathsf{G}_x, \mathfrak{f})$. If S' is another k-torus with $l \subset A(\mathsf{S}')$, there exists $g \in \mathcal{G}_x$ fixing l so that $g\mathsf{S}g^{-1} = \mathsf{S}'$. That g fixes l means the image of g in $\mathsf{G}_x(\mathfrak{f})$ lies in $\mathsf{P}(l)$ and conjugates S_x to S'_x. So, $\theta_{\mathsf{S}_x}(l) = \theta_{\mathsf{S}'_x}(l)$. We drop the subscript and merely write $\theta : \mathcal{L}(x) \to \mathcal{L}(\mathsf{G}_x, \mathfrak{f})$. We leave it to the reader to show that θ is a bijection (see [T: §3.5.4]).

5.3. Suppose $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is a nontrivial involution. The associated action of τ on $\mathcal{B}$ is a polysimplicial isometry. It is elementary $\mathcal{B}^\tau$ is nonempty. The set $\mathcal{B}^\tau$ is also obviously convex and so it has a well-defined dimension. Let m denote its dimension. Suppose $x \in \mathcal{B}^\tau$. The following are elementary:

i) τ acts on $\mathrm{Star}(x)$ and $\mathcal{L}(x)$.
ii) The parahoric subgroup fixing x is stable under τ.
iii) τ induces an involution of the $\mathfrak{f}$-group G_x. For ease of notation, we also denote this induced involution by τ.

iv) $\theta \circ \tau = \tau \circ \theta$.

Let $\mathrm{rk}_{\mathfrak{f}}((\mathsf{G}_x)^\tau)$ denote the (reductive) $\mathfrak{f}$-rank of $(\mathsf{G}_x)^\tau$. The dimension of $\mathcal{L}(\mathsf{G}_x, \mathfrak{f})^\tau$ equals $\mathrm{rk}_{\mathfrak{f}}((\mathsf{G}_x)^\tau) - 1$. Since we have seen $\mathcal{L}(\mathsf{G}_x, \mathfrak{f})^\tau$ can be identified with the rays in $\mathrm{Star}(x)^\tau$ starting from x, we also have $\dim(\mathcal{L}(\mathsf{G}_x, \mathfrak{f})^\tau)$ equals $m-1$, and thus $m = \mathrm{rk}_{\mathfrak{f}}((\mathsf{G}_x)^\tau)$.

5.4. Proposition. *Suppose $x \in \mathcal{B}^\tau$. Then, there exists a τ-invariant maximal k-split torus $\mathsf{S} \subset \mathsf{G}$ so that*

i) $x \in A(\mathsf{S})^\tau$.

ii) $\dim(A(\mathsf{S})^\tau) = m$.

5.5. *Preliminary remarks on the proof of Proposition 5.4.* We begin by outlining our proof. By Proposition 3.4.1, there exists a τ-invariant maximal $\mathfrak{f}$-split torus T_0 in G_x, so that T_0^τ is a maximal $\mathfrak{f}$-split torus in $(\mathsf{G}_x)^\tau$. In particular, $\dim_{\mathfrak{f}}(\mathsf{T}_0^\tau) = m$. Let T be a maximal k-split torus so that $x \in A(\mathsf{T})$ and $\mathsf{T}_0 = \mathsf{T}_x$. The torus T is not necessarily τ-invariant. However, the τ-invariance of T_0 implies that $A(\mathsf{T}) \cap \mathrm{Star}(x)$ is τ-invariant. Furthermore, $\dim(A(\mathsf{T}) \cap \mathrm{Star}(x))^\tau = m$.

If $\mathcal{A}$ is an apartment containing $A(\mathsf{T}, k) \cap \mathrm{Star}(x)$, set

$$\mathcal{A}_\tau := \{\, x \in \mathcal{A} \mid \tau(x) \in \mathcal{A} \,\}.$$

It is elementary that $\mathcal{A}_\tau$ is τ-invariant, convex and a union of (closed) chambers of $\mathcal{A}$. To prove Proposition 5.4, our goal is to begin with the apartment $\mathcal{A}_0 := A(\mathsf{T})$ and show there is a sequence of apartments $\mathcal{A}_i$ containing $A(\mathsf{T}) \cap \mathrm{Star}(x)$ with the property that the sets $(\mathcal{A}_i)_\tau$ increase and converge to a τ-invariant apartment A. The k-torus S corresponding to A satisfies the conclusions i) and ii) of Proposition 5.4.

If A is an apartment and $E \subset A$ is a face of a chamber, let H_E denote the affine root hyperplane in A which contains E. Suppose E' is another (possibly the same) face of H_E. The intersection of all apartments containing the two faces E and E' is a convex $(r-1)$-dimensional set which we denote by $\mathcal{C}(E, E')$. Obviously, $\mathcal{C}(E, E')$ is contained in H_E. Define $\mathcal{K} := \mathcal{K}(E, E')$ to be the collection of the sets Ω satisfying the following properties:

i) Ω contains $\mathcal{C}(E, E')$,

ii) Ω is convex and a union of chambers,

iii) Ω is minimal (under inclusion) among all sets satisfying i) and ii).

5.5.1. Lemma. *Suppose H_ψ is an affine root hyperplane in an apartment A and E, E' are two faces in H_ψ.*

i) If D is a chamber with E as a face, then there exists $\Omega \in \mathcal{K}(E, E')$ such that $D \subset \Omega$.

ii) If $\Omega_1, \Omega_2 \in \mathcal{K}(E, E')$ share a chamber D, then $\Omega_1 = \Omega_2$.

Proof. We first consider the situation when $\mathcal{C}(E, E')$ and D all belong to the apartment A. The existence of an $\Omega \in \mathcal{K}$ reduces to the existence of a union of chambers in A which contains $\mathcal{C}(E, E')$, is convex and is minimal under inclusions. In the latter situation, if we take Ω to be the intersection of all half apartments in A which contain D and $\mathcal{C}(E,E')$, then clearly Ω is minimal under inclusions. This implies existence. Uniqueness of Ω is a consequence of the property that Ω must contain the union of all chambers (in A) whose interiors meet any geodesic segment with one endpoint in the interior of the chamber D and the other endpoint in $\mathcal{C}(E,E')$. To treat the general situation, we observe that the affine root group $\mathcal{U}_\psi$ acts on the set $\mathcal{K}$. Indeed, the $\mathcal{U}_\psi$ action on the chambers containing E has two orbits (one for each of the two chambers in A containing E). Parts i) and ii) follow. $\square$

5.5.2. Corollary. *Let $p = char(k) \neq 2$. Then, $|\mathcal{K}|$ has the form $1 + p^t$.*

Proof. Let C_1 and C_2 denote the two chambers of A containing the face E and let $\Omega(C_1)$ and $\Omega(C_2)$ be the elements of $\mathcal{K}$ which contain C_1 and C_2 respectively. As stated in the proof of Lemma 5.5.1, the affine root group $\mathcal{U}_\psi$ acts on $\mathcal{K}$ with two orbits – the orbits of $\Omega(C_1)$ and $\Omega(C_2)$. It is clear $\Omega(C_1)$ is a singleton orbit and the size of the $\mathcal{U}_\psi$-orbit of $\Omega(C_2)$ is a power of p. The corollary follows. $\square$

5.6. *Completion of the proof of Proposition 5.4.* We define the sequence $\mathcal{A}_i$ inductively. If at any stage $(\mathcal{A}_i)_\tau = \mathcal{A}_i$, i.e. the apartment $\mathcal{A}_i$ is τ-invariant, we take A to be $\mathcal{A}_i$ and we are done. Therefore, we shall always suppose $(\mathcal{A}_i)_\tau \neq \mathcal{A}_i$. We can choose two chambers $C, D \subset \mathcal{A}_i$ so that:

i) $F = C \cap D$ is a face of C (hence D).
ii) $C \subset (\mathcal{A}_i)_\tau$, $D \not\subseteq (\mathcal{A}_i)_\tau$.

We descriptively refer to the face F as a *boundary face* of $(\mathcal{A}_i)_\tau$. We can and do assume (for the purpose of causing convergence in the compact open topology) that the distance of the boundary face F to the point x is minimal for all boundary faces of $(\mathcal{A}_i)_\tau$. Denote as H_F the affine root hyperplane in $\mathcal{A}_i$ determined by F. Let J denote the half apartment of $\mathcal{A}_i \backslash H_F$ containing C. Our proof divides into the following two cases: Case i) $H_F \neq H_{\tau(F)}$. Case ii) $H_F = H_{\tau(F)}$.

Case i) $H_F \neq H_{\tau(F)}$. Let J' denote (open) half apartment of $\mathcal{A}_i \backslash H_{\tau(F)}$ containing $\tau(C)$. The set $(\mathcal{A}_i)_\tau$ is contained in $J \cap J'$. Let ψ denote affine root with the property $H_{\tau(F)} = H_\psi$ and the property that the affine root group $\mathcal{U}_\psi$ fixes $\tau(C)$. The group $\mathcal{U}_\psi$ fixes the half apartment J' and permutes the chambers in the interior of $\mathrm{Star}(\tau(F)) \backslash \tau(C)$ transitively. Choose $u \in \mathcal{U}_\psi$ so that $\tau(D) \subset u\mathcal{A}_i$. Our hypothesis that $H_F \neq H_{\tau(F)}$ implies D and C, hence D and $(\mathcal{A}_i)_\tau$, lie on the same side of $H_{\tau(F)}$. We conclude from this that u fixes D and $(\mathcal{A}_i)_\tau$. Set $\mathcal{A}_{i+1} := u\mathcal{A}_i$. Then $(\mathcal{A}_i)_\tau \cup D \subset \mathcal{A}_{i+1}$ and indeed $(\mathcal{A}_i)_\tau \cup D \subset (\mathcal{A}_{i+1})_\tau$.

Case ii) $H_F = H_{\tau(F)}$. Since J contains $(\mathcal{A}_i)_\tau \supset C$ and $(\mathcal{A}_i)_\tau$ is τ-invariant, we have $\tau(C) \subset J$. The two sets $\mathcal{C}(F,\tau(F))$ and $\mathcal{K}(F,\tau(F))$ are τ-invariant. Since we are assuming the residual characteristic of k is odd, we conclude from Corollary 5.5.2 that $|\mathcal{K}(F,\tau(F))|$ is even. The unique element of $\mathcal{K}(F,\tau(F))$ containing C is obviously τ-invariant; hence there must also exists $\Omega \in \mathcal{K}(F,\tau(F))$ which is τ-invariant and does not contain C. Choose

$u \in \mathcal{U}_\psi$ so that $uD \subset \Omega$ and set $\mathcal{A}_{i+1} = u\mathcal{A}_i$. Then, $(\mathcal{A}_{i+1})_\tau \supset (\mathcal{A}_i)_\tau \cup \Omega$. This completes our induction step.

As already mentioned, in our construction of the sequence $\mathcal{A}_i$, at each stage, if there exists boundary faces on $(\mathcal{A}_i)_\tau$ we choose a boundary face F whose distance to x is minimal among boundary faces. Then, $F \subset (\mathcal{A}_{i+1})_\tau$ and is no longer a boundary face. It follows the sequence of apartments $\mathcal{A}_i$ converge in the compact open topology to an apartment A which is τ-invariant. Hence the maximal k-split torus S associated to A is τ-invariant, and clearly i) $x \in A$ and ii) $\dim(A^\tau) = m$. $\square$

5.7. Corollary.

i) Suppose G *is a reductive* k*-group,* $\tau \in Aut_k(\mathsf{G})$ *is an involution and* T *is a* τ*-invariant* k*-split torus. Then, there exists a* τ*-invariant maximal* k*-split torus* S *containing* T*.*

ii) Suppose G *is a semisimple* k*-group,* $\tau \in Aut_k(\mathsf{G})$ *is an involution and* $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$*. Then,* $\dim(\mathcal{B}^\tau) = \mathrm{rk}_k(\mathsf{G}')$*.*

Proof. To prove i), let $\mathsf{L} := \mathsf{L}(\mathsf{T})$ denote the Levi k-subgroup which is the centralizer of T, and let $\mathsf{Z} := \mathsf{Z}(\mathsf{L})$ denote the center of L. The quotient $\mathsf{M} := \mathsf{L}/\mathsf{Z}$ is a (semisimple) k-group and τ induces a k-involution in $\mathrm{Aut}_k(\mathsf{M})$, which we shall for ease of notation also denote by τ. By Proposition 5.4, there is a τ-invariant maximal k-split torus $\overline{\mathsf{S}}$ in M. Let C denote the preimage in L of $\overline{\mathsf{S}}$. We can take S to be the maximal k-split torus of C.

To prove ii), suppose T is a maximal k-split torus of G^τ. By i), there exists a τ-invariant maximal k-split torus S of G containing T. We have

$$\dim(\mathcal{B}^\tau) \geq \dim(A(\mathsf{S})^\tau) \geq \dim(\mathsf{T}) = \mathrm{rk}_k(\mathsf{G}').$$

Also, by Proposition 5.4, there exists a τ-invariant maximal k-split torus V so that

$$\dim(\mathcal{B}^\tau) = \dim A(\mathsf{V})^\tau = \dim_k(\mathsf{V}^\tau) \leq \mathrm{rk}_k(\mathsf{G}')$$

So $\dim(\mathcal{B}^\tau) = \mathrm{rk}_k(\mathsf{G}')$. $\square$

6. Involutions and affine buildings II

6.1. We continue with the assumptions of section 5: k is a nonarchimedean local field with ring of integers $\mathfrak{o}$ and residue field $\mathfrak{f}$ of odd characteristic, G is a connected semisimple group defined over k and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is a nontrivial involution. Let G' denote the connected reductive group $(\mathsf{G}^\tau)^\circ$ and set $\mathcal{G}' := \mathsf{G}'(k) \subset \mathcal{G} = \mathsf{G}(k)$. If $\mathcal{H}$ is a subgroup of $\mathcal{G}$, let $\mathcal{C}_\mathcal{G}(\mathcal{H})$ and $\mathcal{N}_\mathcal{G}(\mathcal{H})$ denote the centralizer and normalizer of $\mathcal{H}$ respectively. We use similar notation for subgroups of $\mathcal{G}^\tau$ and $\mathcal{G}'$.

The main result in this section is Theorem 6.7.3. It states under the assumption G and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ are as in section 4 and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple that $\mathcal{B}' := \mathcal{B}(\mathsf{G}', k) = \mathcal{B}(\mathsf{G}')$ identifies naturally with $\mathcal{B}(\mathsf{G})^\tau$. We outline our plan to accomplish this.

i) Suppose T is a maximal k-split torus of G'. Set $\mathcal{T} := \mathsf{T}(k)$. Let $A(\mathsf{T}, \mathsf{G}')$ denote the apartment in $\mathcal{B}(\mathsf{G}')$ associated to T. We find a canonical affine subspace $A(\mathsf{T}, \mathsf{G})$ of $\mathcal{B}^\tau$ which we eventually identify with the apartment $A(\mathsf{T}, \mathsf{G}')$.

ii) We show the restriction of the action of $\mathcal{G}$ on $\mathcal{B}$ to the group $\mathcal{N}_{\mathcal{G}'}(\mathcal{T}) \subset \mathcal{G}$ yields an action on the affine subspace $A(\mathsf{T}, \mathsf{G})$ which is equivalent to the action of $\mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ on $A(\mathsf{T}, \mathsf{G}')$ coming from the action of $\mathcal{G}'$ on $\mathcal{B}'$.

iii) Suppose $x \in A(\mathsf{T}, \mathsf{G})$ and S is a τ-stable maximal k-split torus of G containing T. Set $\mathcal{S} := \mathsf{S}(k)$. Let $\mathcal{V}_x$ (resp. $\mathcal{U}_x$) denote the subgroup generated by the affine $\mathcal{T}$-root groups (resp. affine $\mathcal{S}$-root groups) of $\mathcal{G}'$ (resp. $\mathcal{G}$) fixing the point x. We show $\mathcal{V}_x \subset \mathcal{U}_x \cap \mathcal{G}'$.

We then identify $\mathcal{B}^\tau$ with $\mathcal{B}'$ so that the restriction action of $\mathcal{G}' \subset \mathcal{G}$ on $\mathcal{B}^\tau$ is equivalent to its action on $\mathcal{B}'$.

6.2. Although we shall eventually assume G and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ are as in section 4, our preliminary results (Propositions 6.2.1 and 6.3.1) hold in the generality of G an arbitrary semisimple k-group and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ an arbitrary involution.

6.2.1. Proposition. *Suppose* T *is a maximal* k*-split torus of* G'*.*

i) There exists a τ*-invariant maximal* k*-split torus* S *of* G *containing* T*.*

ii) If S_1 *and* S_2 *are two* τ*-invariant maximal* k*-split tori of* G *containing* T*, then*

$$A(\mathsf{S}_1)^\tau = A(\mathsf{S}_2)^\tau.$$

Proof. Assertion i) is Corollary 5.7 i). To prove assertion ii), we have by Corollary 5.7 ii) that $\dim(\mathcal{B}(\mathsf{G})^\tau) = \mathrm{rk}_k(\mathsf{G}') = \dim(A(\mathsf{S}_i)^\tau)$. Let L denote the Levi k-subgroup $\mathsf{C}_{\mathsf{G}}(\mathsf{T})$. Set

$$\mathcal{B}_{\mathsf{L}} := \text{ union of the apartments } A(\mathsf{S}) \subset \mathcal{B}(\mathsf{G}) \text{ with } \mathsf{S} \text{ containing } \mathsf{T}.$$

It is obvious $\mathcal{B}_{\mathsf{L}}$ is convex, τ-stable. The dimension of the convex set $(\mathcal{B}_{\mathsf{L}})^\tau$ satisfies

$$\dim(\mathcal{B}(\mathsf{G})^\tau) \geq \dim((\mathcal{B}_{\mathsf{L}})^\tau) \geq \mathrm{rk}_k(\mathsf{G}') = \dim(\mathcal{B}(\mathsf{G})^\tau),$$

so $\dim((\mathcal{B}_{\mathsf{L}})^\tau) = \mathrm{rk}_k(\mathsf{G}')$. The vector space $V(\mathsf{T}) := X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ acts naturally on $(\mathcal{B}_{\mathsf{L}})^\tau$. Let $(\mathcal{B}_{\mathsf{L}})^\tau / V(\mathsf{T})$ denote the quotient space. We have

$$\dim((\mathcal{B}_{\mathsf{L}})^\tau) = \mathrm{rk}_k(\mathsf{T}) + \dim((\mathcal{B}_{\mathsf{L}})^\tau / V(\mathsf{T})). \tag{6.2.2}$$

So, $\dim((\mathcal{B}_{\mathsf{L}})^\tau / V(\mathsf{T})) = 0$ and therefore the convex set $(\mathcal{B}_{\mathsf{L}})^\tau / V(\mathsf{T})$ must be a singleton point. This in particular means $(A(\mathsf{S}_1))^\tau = (A(\mathsf{S}_2))^\tau$ $\square$

As a consequence of this proposition, any maximal k-split torus T of G' determines a canonical affine subspace of $\mathcal{B}(\mathsf{G})^\tau$ of dimension $\mathrm{rk}_k(\mathsf{G}')$. We denote this set as $A(\mathsf{T}, \mathsf{G})$.

6.3. The maximal k-split tori of G^τ and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ obviously coincide. Suppose T is such a maximal k-split torus of G' and $\mathcal{T} := \mathsf{T}(k)$. Let $\mathcal{N}_{\mathcal{G}}(\mathcal{T})$ (resp. $\mathcal{C}_{\mathcal{G}}(\mathcal{T})$) denote the normalizer (resp. centralizer) of $\mathcal{T}$ in $\mathcal{G}$. All these groups are τ-stable. The group $\mathcal{N}_{\mathcal{G}}(\mathcal{T})$ acts on the space $\mathcal{B}_{\mathsf{L}}$ of (6.2.2); thus $\mathcal{N}_{\mathcal{G}}(\mathcal{T})^\tau = \mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T})$ acts on $A(\mathsf{T}, \mathsf{G})$. The groups $\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T}) = \mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau$ and $\mathcal{C}_{\mathcal{G}'}(\mathcal{T}) \subset \mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})$ are compact modulo $\mathcal{T}$ and so they have unique maximal bounded subgroups $(\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T}))^b$ and $(\mathcal{C}_{\mathcal{G}'}(\mathcal{T}))^b$ respectively.

6.3.1. Proposition. *Suppose* T *is a maximal* k*-split torus of* G'*. The maximal bounded subgroup* $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ *acts trivially on* $A(\mathsf{T},\mathsf{G})$*.*

Proof. The compact group $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ acts isometrically on the Euclidean space $A(\mathsf{T},\mathsf{G})$; hence there is a fixed point $x \in A(\mathsf{T},\mathsf{G})$. Any $s \in \mathcal{N} = \mathcal{N}_{\mathcal{G}}(\mathcal{T})^\tau$ normalizes $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ and hence $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ must fix sx. This means $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ must fix all the points $\mathcal{N}x$ and in particular all the points $\mathcal{T}x$. But $\mathcal{T}$ acts on $A(\mathsf{T},\mathsf{G})$ cocompactly, so we conclude $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ must acts trivially on the points of a cocompact lattice of $A(\mathsf{T},\mathsf{G})$ and thus $(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ must fix $A(\mathsf{T},\mathsf{G})$. □

6.4. It follows immediately from Proposition 6.3.1 that the action of $\mathcal{N}_{\mathcal{G}}(\mathcal{T})^\tau$ on $A(\mathsf{T},\mathsf{G})$ factors to an action of $\mathcal{N}_{\mathcal{G}}(\mathcal{T})^\tau/(\mathcal{C}_{\mathcal{G}}(\mathcal{T})^\tau)^b$ on $A(\mathsf{T},\mathsf{G})$. Our next step is to establish the compatibility of the action of $\mathcal{N}_{\mathcal{G}}(\mathcal{T})^\tau$ on $A(\mathsf{T},\mathsf{G})$ with the action of $\mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ on $A(\mathsf{T},\mathsf{G}')$. Due to present limitations in our proof, we shall only be able to accomplish this when G is a special linear group and $\tau \in \mathrm{Aut}_k(\mathsf{G})$ is an involution as in section 4.

As a first step in this direction, we consider the action of $\mathcal{C}_{\mathcal{G}'}(\mathcal{T})$ on the affine spaces $A(\mathsf{T},\mathsf{G}')$ and $A(\mathsf{T},\mathsf{G})$. Define the homomorphism $\nu : \mathcal{C}_{\mathcal{G}'}(\mathcal{T}) \to X_*(\mathsf{T})\otimes_{\mathbb{Z}}\mathbb{R} \subset X_*(\mathsf{S})\otimes_{\mathbb{Z}}\mathbb{R}$ as in [SS: §1.1]. Then, the action of an element $v \in \mathcal{T}$ on both $A(\mathsf{T},\mathsf{G}')$ and $A(\mathsf{T},\mathsf{G}) \subset A(\mathsf{S},\mathsf{G})$ is a translation

$$tx = x + \nu(v).$$

6.4.1. Suppose now that Γ, G and $\tau \in \mathrm{Aut}_k(\Gamma)$ are as defined in section 4, more specifically by (4.4.8) and (4.4.2). Recall also $\mathcal{G} = \Gamma(k) = \mathrm{GL}_D(V)$ where $V = D^n$. Let $\langle\, , \rangle$ be the form (4.4.4). Set $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$.

6.4.2 Lemma. *In the notation of (6.4.1), suppose* T *is a maximal* k*-split torus in* G' *and suppose* $\tilde{\mathsf{S}}$ *is a* τ*-stable maximal* k*-split torus in* Γ *containing* T*. Let* $\tilde{\mathcal{S}}$ *be* $\tilde{\mathsf{S}}(k)$*. Then*

$$(\mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T}) \cap \mathcal{N}_{\mathcal{G}}(\tilde{\mathcal{S}}))\, \mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b = \mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T}). \tag{6.4.3}$$

Proof. Let T and $\tilde{\mathsf{S}}$ be as in the lemma. Let m be the Witt index of $\langle\, , \rangle$ and $m_0 = n-2m$. By the Witt basis theorem, we have a basis $\{e_i,\, f_j \mid i = \pm 1, \cdots \pm m,\, j = 1, \cdots m_0\}$ which consists of eigenspaces of $\mathcal{T}$ with $\langle e_i, e_{-j}\rangle = \delta_{ij} = \epsilon\langle e_{-j}, e_i\rangle$ for $i > 0$, $\langle e_i, f_j\rangle = 0$, and $\langle f_i, f_j\rangle = 0$ for $i \neq j$. Let V_s be a subspace of V generated by $\{e_i \mid i = \pm 1, \cdots, \pm m\}$ over D and let V_0 be generated by $\{f_j \mid j = 1, \cdots, m_0\}$. Then $V = V_s \oplus V_0$ is an orthogonal decomposition with respect to $\langle\, , \rangle$. We can naturally identify $\mathrm{GL}_D(V_s)$, $\mathrm{GL}_D(V_0)$ and $\mathrm{GL}_D(V_s) \times \mathrm{GL}(V_0)$ as subgroups of $\mathcal{G} = \mathrm{GL}_D(V)$. Let $\mathcal{G}(V_s) = \mathcal{G}^\tau \cap \mathrm{GL}_D(V_s)$ and $\mathcal{G}(V_0) = \mathcal{G}^\tau \cap \mathrm{GL}_D(V_0)$. Then we have a natural embedding of $\mathcal{G}(V_s) \times \mathcal{G}(V_0)$ into $\mathcal{G}^\tau$ and of $\mathrm{GL}_D(V_s) \times \mathrm{GL}_D(V_0)$ into $\mathcal{G}$. Via these embeddings, we have $\mathcal{T} \subset \mathcal{G}(V_s) \times \mathcal{G}(V_0)$ and $\tilde{\mathcal{S}} \subset \mathrm{GL}_D(V_s) \times \mathrm{GL}_D(V_0)$. Moreover, $\mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T})$ can be decomposed into a product of $\mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T}) \cap \mathcal{G}(V_s)$ and $\mathcal{G}(V_0)$. In (6.4.3), $\subset$ is obvious. Since $(\mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T}) \cap \mathcal{N}_{\mathcal{G}}(\tilde{\mathcal{S}})) \supset (\mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T}) \cap \mathcal{G}(V_s))$ and $\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b \supset \mathcal{G}(V_0)$, $\supset$ also follows. □

Since $\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b$ is not contained in $\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b$ in general, it is not obvious that there is a well defined monomorphism $\mathcal{N}_{\mathcal{G}'}(\mathcal{T})/\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b \longrightarrow \mathcal{N}_{\mathcal{G}}(\mathcal{S})/\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b$. In the following corollary, we define such a map by Lemma 6.4.2. Note that there are four different groups $\mathcal{G}'$, $\mathcal{G}$, $\mathcal{G}^\tau$ and $\mathcal{G}$ involved in the proof.

6.4.4. Corollary. *We keep the notation from (6.4.1) and let* $\mathsf{S} = \tilde{\mathsf{S}} \cap \mathsf{G}$. *If* $n\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b$ *is a coset of* $\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b$ *in* $\mathcal{N}_{\mathcal{G}'}(\mathcal{T})$, *let* $n' \in n\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b \cap \mathcal{N}_{\mathcal{G}}(\mathcal{S})$. *The element* n' *is unique modulo the subgroup* $\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b$ *and the map*

$$\begin{aligned}\phi : \mathcal{N}_{\mathcal{G}'}(\mathcal{T})/\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b &\longrightarrow \mathcal{N}_{\mathcal{G}}(\mathcal{S})/\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b \\ \phi(nC_{\mathcal{G}'}(\mathcal{T})^b) &= n'C_{\mathcal{G}}(\mathcal{S})^b\end{aligned}$$

is a monomorphism of $\mathcal{N}_{\mathcal{G}'}(\mathcal{T})/\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b$ *into* $\mathcal{N}_{\mathcal{G}}(\mathcal{S})/\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b$.

Proof. Observe that $\mathcal{N}_{\mathcal{G}}(\mathcal{S}) = \mathcal{N}_{\mathcal{G}}(\tilde{\mathcal{S}})$ and that there are natural monomorphisms

$$\begin{aligned}\phi_1 &: \mathcal{N}_{\mathcal{G}'}(\mathcal{T})/\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b \longrightarrow \mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T})/\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b, \\ \phi_2 &: \mathcal{N}_{\mathcal{G}}(\mathcal{S})/\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b \longrightarrow \mathcal{N}_{\mathcal{G}}(\tilde{\mathcal{S}})/\mathcal{C}_{\mathcal{G}}(\tilde{\mathcal{S}})^b.\end{aligned}$$

By the above lemma, we can also define a monomorphism

$$\phi_3 : \mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T})/\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b \longrightarrow \mathcal{N}_{\mathcal{G}}(\tilde{\mathcal{S}})/\mathcal{C}_{\mathcal{G}}(\tilde{\mathcal{S}})^b$$

such that for $m\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b$, a coset of $\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b$ in $\mathcal{N}_{\mathcal{G}^\tau}(\mathcal{T})$ and $m' \in m\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b \cap \mathcal{N}_{\mathcal{G}}(\tilde{\mathcal{S}})$, $\phi_3(m\mathcal{C}_{\mathcal{G}^\tau}(\mathcal{T})^b) = m'\mathcal{C}_{\mathcal{G}}(\tilde{\mathcal{S}})^b$. Then the image of $\phi_3 \circ \phi_1$ factors through $\mathcal{N}_{\mathcal{G}}(\mathcal{S})/\mathcal{C}_{\mathcal{G}}(\mathcal{S})^b$, that is, we have $\phi_3 \circ \phi_1 = \phi_2 \circ \phi$ and the corollary is proved. □

6.5. We introduce some notation. Suppose T is a maximal k-split torus in G'. If Y is a subset of the apartment $A(\mathsf{T}, \mathsf{G}')$, and $w \in X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$, set

$$\begin{aligned}Y + w &:= \{\, x + w \mid x \in Y \,\} \\ Y(w) &:= \{\, x + tw \mid x \in Y \text{ and } t \geq 0\}\end{aligned}$$

If ψ is a nonconstant affine root on $A(\mathsf{T}, \mathsf{G}')$ with gradient $\alpha \in \Phi(\mathsf{G}', \mathsf{T})$ and H_ψ is the vanishing hyperplane of ψ, then $H_\psi(\alpha)$ is a half space in $A(\mathsf{T}, \mathsf{G}')$. We use similar notation for G.

Suppose $\alpha \in \Phi(\mathsf{G}', \mathsf{T})$ and u is a nonidentity element of the α root group $\mathcal{V}_\alpha \subset \mathcal{G}'$. The unique element in $\mathcal{V}_{-\alpha} u \mathcal{V}_{-\alpha} \cap \mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ is expressible [BT:§4.1.5.2] as $m(u) = vuv$, where $v \in \mathcal{V}_{-\alpha}$. The element $m(u)$ acts on $A(\mathsf{T}, \mathsf{G}')$ as a reflection across a hyperplane $H = H_u$. The elements v and u, as elements in $\mathcal{G}'$, fix the half-spaces $H(-\alpha)$ and $H(\alpha)$ of $A(\mathsf{T}, \mathsf{G}')$ respectively. We compare this to the situation where we view v and u as elements of $\mathcal{G}$ acting on $\mathcal{B}(\mathsf{G})$. It follows from the Witt basis in Lemma 6.4.2 and Corollary 6.4.4, that $m(u)$ is reflection across a hyperplane h_u in $A(\mathsf{T},\mathsf{G})$. We intend to identify $A(\mathsf{T},\mathsf{G})$ and $A(\mathsf{T},\mathsf{G}')$ so that the hyperplane h_u is identified with the hyperplane $H = H_u$. As a step towards this goal, we now show that in $A(\mathsf{T},\mathsf{G})$ the elements v and u fix the affine half-space $h_u(-\alpha)$ and $h_u(\alpha)$ respectively.

Choose a τ-invariant maximal k-split torus S in G containing T. When viewed as an element in $\mathcal{G}$, u is a product of elements in affine root groups $\mathcal{U}_\phi$ (of G with respect to S) satisfying

$$\operatorname{grad}(\phi|_{A(\mathsf{T},\mathsf{G})}) \text{ is a multiple of } \alpha. \tag{6.5.1}$$

Suppose ϕ satisfies (6.5.1). Then, the affine root group $\mathcal{U}_\phi$ fixes the half-space $H_\phi(\alpha)$ and so fixes the affine half-subspace $h_u(\alpha) + R\alpha$ provided $R \geq 0$ is sufficiently large. Since u lies in a product of affine root groups satisfying (6.5.1), it follows there exists an $R_u \geq 0$ so that u fixes the affine half-subspace $h_u(\alpha) + R_u\alpha$. This in combination with convexity of the u-fixed point $A(\mathsf{T},\mathsf{G})^u$ implies either $A(\mathsf{T},\mathsf{G})^u$ equals $A(\mathsf{T},\mathsf{G})$ or is a half-space. We shall see soon that $A(\mathsf{T},\mathsf{G})^u$ is a half-space.

In a similar fashion, there is an $R_v \geq 0$ so that v fixes the affine half-subspaces $h_u(-\alpha) - R_v\alpha$, and the fixed point set is either a half-space or all of $A(\mathsf{T},\mathsf{G})$. Suppose either $A(\mathsf{T},\mathsf{G})^u$ or $A(\mathsf{T},\mathsf{G})^v$ is all of $A(\mathsf{T},\mathsf{G})$. From $m(u) = vuv$, we conclude that $m(u)$ must fix a half-space of $A(\mathsf{T},\mathsf{G})$. This is clearly a contradiction to the action of $m(u)$ being a reflection across the hyperplane h_u. Hence u and v fix half-spaces in $A(\mathsf{T},\mathsf{G})$. Let $b(u)$ and $b(v)$ denote the boundary hyperplanes of the two half-spaces $A(\mathsf{T},\mathsf{G})^u$ and $A(\mathsf{T},\mathsf{G})^v$. Heuristically, the action of u on $A(\mathsf{T},\mathsf{G})$ is that it fixes pointwise the half-space $A(\mathsf{T},\mathsf{G})^u$ and 'folds along the hyperplane $b(u)$' the closure of the complementary half-space 'away' from $A(\mathsf{T},\mathsf{G})$. The situation for v is similar.

If the intersection of the two half-spaces $A(\mathsf{T},\mathsf{G})^u$ and $A(\mathsf{T},\mathsf{G})^v$ contains an open neighborhood U, then U must be fixed by the reflection $m(u)$, an impossibility. Hence, either the intersection of the two half-spaces is empty or a hyperplane, which subsequently must be h_u. To rule out the intersection being empty, we argue by contradiction. Assume the intersection is empty. Choose a point x in the hyperplane $b(v)$. Then, the empty intersection hypothesis implies

i) $A(\mathsf{T},\mathsf{G})^u$ is contained in $A(\mathsf{T},\mathsf{G}) - A(\mathsf{T},\mathsf{G})^v$,
ii) the geodesic segment $[x, ux]$ meets $A(\mathsf{T},\mathsf{G})^v$ precisely at the point x,
iii) there exists y in the open interval (x, ux) so that $(x, y]$ lies in $A(\mathsf{T},\mathsf{G}) - A(\mathsf{T},\mathsf{G})^v$.

Now, $v[x, ux] = [vx, m(u)x]$ lies in $A(\mathsf{T},\mathsf{G})$ since its two endpoints lie in $A(\mathsf{T},\mathsf{G})$. We conclude that the geodesic interval $[x, y]$ is sent by v to another geodesic interval inside $A(\mathsf{T},\mathsf{G})$. But, since $[x, y]$ meets $A(\mathsf{T},\mathsf{G})^v$ in the point x, and v folds the half-space which is the closure of $A(\mathsf{T},\mathsf{G}) - A(\mathsf{T},\mathsf{G})^u$ off $A(\mathsf{T},\mathsf{G})$ along the hyperplane $b(u)$, it is impossible for $v[x, y]$ to be in $A(\mathsf{T},\mathsf{G})$. This is a contradiction. We conclude that v and u must fix a common point in $A(\mathsf{T},\mathsf{G})$, from which we deduce that $b(u) = b(v) = h_u$ is the common fixed set of u and v in $A(\mathsf{T},\mathsf{G})$. Thus, v fixes the affine half-subspace $h_u(-\alpha)$, and u fixes $h_u(\alpha)$.

6.6. Let $\iota_\alpha : A(\mathsf{T},\mathsf{G}') \to A(\mathsf{T},\mathsf{G})$ be any isometry, i.e. a $\mathcal{C}_{\mathcal{G}'}(\mathcal{T})$-equivariant map, which takes h_u to H_u. The map ι_α is determined up to a translation by a vector parallel to H_u. Our discussion implies

i) ι_α commutes with the actions of $m(u)$ on $A(\mathsf{T},\mathsf{G}')$ and $A(\mathsf{T},\mathsf{G})$.
ii) The element v fixes the affine half-subspace $H_u(-\alpha) = \iota_\alpha(h_u(-\alpha))$, and u fixes $H_u(\alpha) = \iota_\alpha(h_u(\alpha))$.

The above discussion is for a single root $\alpha \in \Phi(\mathsf{G}',\mathsf{T})$. If we consider all roots, the assumption that G' is semisimple, i.e. $X_*(\mathsf{T}) \otimes_{\mathbb{Z}} \mathbb{R}$ is spanned by the simple roots, implies there is a unique $\mathcal{C}_{\mathcal{G}'}(\mathcal{T})$-equivariant map $\iota : A(\mathsf{T},\mathsf{G}') \to A(\mathsf{T},\mathsf{G})$ so that

i) For any root $\alpha \in \Phi(\mathsf{G}',\mathsf{T})$ and $u \in \mathcal{V}_\alpha \backslash \{1\}$, the map ι takes h_u to H_u and ι commutes with the actions of $m(u) = vuv$ on $A(\mathsf{T},\mathsf{G})$ and $A(\mathsf{T},\mathsf{G}')$.
ii) The element v fixes the affine half-subspace $H_u(-\alpha) = \iota(h_u(-\alpha))$, while u fixes $H_u(\alpha) = \iota(h_u(\alpha))$.

For $x \in A(\mathsf{T},\mathsf{G}')$, set

$$\begin{aligned}\mathcal{V}_x := \text{subgroup of } \mathcal{G}' \text{ generated by all the affine root groups}\\ \mathcal{V}_\psi \text{ with respect to } \mathcal{T} \text{ which fix the point } x\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}_{\iota(x)} := \text{subgroup of } \mathcal{G} \text{ generated by all the affine root groups}\\ \mathcal{U}_\phi \text{ with respect to } \mathcal{S} \text{ which fix the point } \iota(x)\end{aligned}$$

From the above, we see

$$\mathcal{V}_x \subset \mathcal{U}_{\iota(x)}.$$

6.7. We are now ready to establish the existence of an injective map $\pi : \mathcal{B}(\mathsf{G}') \to \mathcal{B}(\mathsf{G})^\tau$. To do this, we choose a maximal k-split torus T of G' and a τ-stable maximal k-split torus S of G which contains T. Recall that any point $z \in \mathcal{B}(\mathsf{G}')$ is expressible as $z = gx$, where $x \in \mathcal{G}'$ and $x \in A(\mathsf{T},\mathsf{G}')$. This expression is not unique, but if $z = hy$ is a second representation of z, with $h \in \mathsf{G}'$ and $y \in A(\mathsf{T},\mathsf{G}')$, then there exists an $n \in \mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ so that i) $y = nx$, and ii) $g^{-1}hn \in \mathcal{V}_x$. Define $\pi : \mathcal{B}(\mathsf{G}') \to \mathcal{B}(\mathsf{G})^\tau$ as follows: If z is expressible as $z = gx$, set

$$\pi(z) = g\iota(x) \tag{6.7.1}$$

Here, $\iota(x) \in A(\mathsf{T},\mathsf{G})$ and the action of g on $\iota(x)$ is by viewing g as being in $\mathcal{G}$.

6.7.2. Proposition. *Let π be defined as in (6.7.1).*
i) π is a well defined map.
ii) π is an injection.

Proof. To prove i), suppose $z = gx$ and $z = hy$ are two expressions for the point z, i.e. $g, h \in \mathcal{G}'$ and $x, y \in A(\mathsf{T},\mathsf{G}')$. Choose $n' \in \mathcal{N}_{\mathcal{G}'}(\mathcal{T})$ so that $y = n'x$, and $g^{-1}hn \in \mathcal{V}_x$. According to

Lemma 6.4.2 and Corollary 6.4.4, we can choose $n \in n'\mathcal{C}_{\mathcal{G}'}(\mathcal{T})^b$ so that $\iota(y) = n\iota(x)$. Since $\mathcal{V}_x \subset \mathcal{U}_{\iota(x)}$, we have $g^{-1}hn \in \mathcal{U}_{\iota(x)}$. So, $\iota(x) = g^{-1}hn\iota(x)$, i.e. $g\iota(x) = hn\iota(x) = h\iota(y)$. Hence, π is well-defined.

To prove ii), suppose $z_1 = gx$ and $z_2 = hy$ map to the same point under π, i.e. $g\iota(x) = h\iota(y)$. The Bruhat decomposition $\mathcal{G}' = \mathcal{V}_x\mathcal{N}_{\mathcal{G}'}(\mathcal{T})\mathcal{V}_y$, allows us to express the element $g^{-1}h$ as

$$g^{-1}h = u_x n^{-1} u_y \text{ with } u_x \in \mathcal{V}_x \subset \mathcal{U}_{\iota(x)}, u_y \in \mathcal{V}_y \subset \mathcal{U}_{\iota(y)} \text{ and } n \in \mathcal{N}_{\mathcal{G}'}(\mathcal{T}).$$

Upon substitution, the equality $\iota(x) = g^{-1}h\iota(y)$ becomes $\iota(x) = u_x n^{-1} u_y \iota(y)$ from which we conclude that $y = nx$ and $g^{-1}hn = u_x n^{-1} u_y n \in \mathcal{V}_x$. So $z_1 = gx = hy = z_2$, and therefore π is injective. □

6.7.3. Theorem. *Suppose k is a nonarchimedean local field with odd residual characteristic. Suppose G and $\tau \in Aut_k(\mathsf{G})$ are as in (4.4.10) and (4.4.2) and $\mathsf{G}' := (\mathsf{G}^\tau)^\circ$ is semisimple. Then π is a $\mathsf{G}'(k)$-equivariant isometry of $\mathcal{B}(\mathsf{G}')$ onto $\mathcal{B}(\mathsf{G})^\tau$.*

Proof. Normalize distance on $A(\mathsf{T},\mathsf{G})$ so that the map $\pi|_{A(\mathsf{T},\mathsf{G}')} : A(\mathsf{T},\mathsf{G}') \to A(\mathsf{T},\mathsf{G})$ is an isometry. For $g \in \mathcal{G}'$, and $z \in \mathcal{B}(\mathcal{G}')$, we have $g\pi(z) = \pi(gz)$. Given any two points $z_1, z_2 \in \mathcal{B}(\mathcal{G}')$, there exist points $x_1, x_2 \in A(\mathsf{T},\mathsf{G}')$ and a $g \in \mathcal{G}'$ so that $z_i = gx_i$, thus $\pi(z_i) = \pi(gx_i) = g\pi(x_i)$. Then, $\mathrm{dist}(\pi(z_1),\pi(z_2)) = \mathrm{dist}(\pi(x_1),\pi(x_2)) = \mathrm{dist}(x_1,x_2) = \mathrm{dist}(z_1,z_2)$. So π is an isometry. The final assertion that π is onto is Proposition 5.4 and Corollary 5.7. □

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