JACQUET FUNCTORS AND UNREFINED MINIMAL K-TYPES

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Abstract

The notion of an unrefined minimal K-type is extended to an arbitrary reductive group over a non archimedean local field. This allows one to define the depth of a representation. The relationship between unrefined minimal K-types and the functors of Jacquet is determined. Analogues of fundamental results Borel are proved for representations of depth zero.

1. Introduction

Suppose G is an absolutely quasi-simple, simply connected algebraic group defined over a nonarchimedean local field k and $\mathcal{G} = G(k)$ is the group of its k-rational points. Given a point x in the Bruhat–Tits building of $\mathcal{G}$, the isotropy subgroup of the point x, i.e. $\mathcal{G}_x = \{g \in \mathcal{G} \mid g \cdot x = x\}$, is a parahoric subgroup of $\mathcal{G}$. The authors defined in [MP] natural filtration subgroups of $\mathcal{G}_x$ denoted by $\mathcal{P}_{x,r}$ there, and to be denoted by $\mathcal{G}_{x,r}$ in this paper, and introduced the notion of unrefined minimal K-types for irreducible admissible representations of $\mathcal{G}$. Given any admissible (smooth) irreducible representation π of $\mathcal{G}$ on a complex vector space V_π, as the main result in [MP], the authors showed (when the characteristic of k is zero) that there is a nonnegative rational number $r = \varrho(\pi)$ such that

(1) For some x in the building, the space $V_\pi^{\mathcal{G}_{x,r^+}}$ of $\mathcal{G}_{x,r^+}$-fixed vectors is nonzero and r is the minimal nonnegative number for which this occurs.

(2) For any y in the building with $V_\pi^{\mathcal{G}_{y,r^+}} \neq \{0\}$, the representation τ of $\mathcal{G}_{y,r}$ on $V_\pi^{\mathcal{G}_{y,r^+}}$ contains an unrefined minimal K-type.

The rational number $\varrho(\pi)$ is called the *depth* of the representation π. Furthermore, any two unrefined minimal K-types occurring in π are closely related via a notion of associativity. The extension of this result to a general reductive group as well as when the characteristic of k is positive is rather straightforward – it is done here in section 3 – but for various applications it is useful to have it written down.

As one such application, it is very natural to consider the relationship between unrefined minimal K-types and parabolic induction. A prototype for what can be expected in this area are results of Borel's [B1]. In the case of the unramified principal series $I(\nu)$, Borel has shown

(1) Any subquotient of an unramified principal series contains a nonzero Iwahori fixed vector.
(2) Any irreducible admissible representation which possesses a nonzero Iwahori fixed vector occurs as a subquotient of an unramified principal series.
(3) Let $\mathcal{P} = \mathcal{M}\mathcal{N}$ be a parabolic subgroup of $\mathcal{G}$ (see 2.1 below) and $\mathcal{G}_x$ an Iwahori subgroup of $\mathcal{G}$ (see 3.1 below) with an Iwahori decomposition with respect to $\mathcal{P}$ and $\mathcal{M}$ – e.g. x a point in the building of $\mathcal{M}$. If (π, V_π) is an irreducible admissible representation with a nonzero Iwahori fixed vector, then the Jacquet functor $J_\mathcal{N}$

$$J_\mathcal{N} : V_\pi^{\mathcal{G}_x} \longrightarrow J_\mathcal{N}(V_\pi)^{\mathcal{M}_x}$$

is an isomorphism.

Borel's result (1) can be reformulated as follows into a statement which allows for generalization to unrefined minimal K-types. The depth of an unramified character ν or more generally that of any irreducible admissible representation possessing a nonzero Iwahori fixed vector is zero. Whence, Borel's result can be restated in a weaker form as saying that the depth $\varrho(\pi)$ of any subquotient π of $I(\nu)$ is equal to the depth $\varrho(\nu)$ of the inducing representation ν. In particular, if $\mathcal{P} = \mathcal{M}\mathcal{N}$ is a parabolic subgroup of $\mathcal{G}$ and σ is an irreducible admissible representation of $\mathcal{M}$, we can naturally generalize the depth zero property of subquotients of an unramified principal series to the assertion that any subquotient π of the induced representation $\mathrm{Ind}_\mathcal{P}^\mathcal{G}\sigma$ has depth $\varrho(\pi)$ equal to the depth $\varrho(\sigma)$ of σ. The proof of this assertion is one of our main results (Theorem 5.2). We also prove a similar result for the subquotients of a Jacquet module (Theorem 4.5).

The 'refinement' of an unrefined minimal K-type $(\mathcal{G}_{x,r}, \chi)$ to a refined minimal K-type is still not well understood and presently there is no precise definition of a refined minimal K-type. However, in the case of depth zero minimal K-types, i.e. unrefined minimal K-types of the form $(\mathcal{G}_x, \sigma)$, where $\mathcal{G}_x$ is a parahoric and σ a cuspidal representation of $\mathsf{M}_x(\mathfrak{f}) = \mathcal{G}_x/\mathcal{G}_{x,0^+}$ inflated to $\mathcal{G}_x$, there is no need for refinement. Therefore, we call an unrefined minimal K-type of depth zero a (refined) minimal K-type. In section 6, we extend Borel's results (1), (2) and (3) to arbitrary depth zero representations.

Many of the results on depth zero representations in section 6, including Proposition 6.7 and Theorem 6.11, were obtained a little earlier by Lawrence Morris. Upon completion of this manuscript, we received the preprint 'Level zero G-types' containing his results. Our results were obtained independently of his work and our proofs appear to us to be conceptually simpler and more direct than his.

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2. Jacquet Functors

2.1. The following notation will be used throughout this paper. k will denote a nonarchimedean local (i.e. locally compact) field of arbitrary characteristic and G will be a connected reductive algebraic group defined over k. The group $G(k)$ of k-rational points of G, with the natural locally compact topology induced from that on k, will be denoted by $\mathcal{G}$. The Lie algebra of G will be denoted by $L(G)$ and its dual by $L(G)^*$. The vector space of k-rational points of $L(G)$ (resp. $L(G)^*$) will be denoted by $\mathfrak{g}$ (resp. $\mathfrak{g}^*$); $\mathfrak{g}$ is a k-Lie algebra and $\mathfrak{g}^* = \mathrm{Hom}_k(\mathfrak{g}, k)$. $L(G)^*$ (resp. $\mathfrak{g}^*$) will be considered as a rational G-module (resp. $\mathcal{G}$-module) under the coadjoint action of G (resp. $\mathcal{G}$).

We recall some basic results on the Jacquet functor. Let (π, V_π) be an admissible representation of $\mathcal{G}$. Given a parabolic k-subgroup $P = MN$ of G, where M is a maximal connected reductive k-subgroup of P and N is the unipotent radical, let $\mathcal{P}$, $\mathcal{M}$ and $\mathcal{N}$ be the groups of k-rational points of P, M and N respectively. We shall say that a subgroup of $\mathcal{G}$ is a parabolic subgroup if it is the group of k-rational points of a parabolic k-subgroup of G. Thus $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$; $\mathcal{M}$ is called a Levi factor (or a Levi subgroup) of $\mathcal{P}$. The Jacquet module $J_{\mathcal{N}}(V_\pi)$ associated to π is the representation of $\mathcal{M}$ on the space of $\mathcal{N}$-coinvariants of V_π. Thus, $J_{\mathcal{N}}(V_\pi) = V_\pi / V_\pi(\mathcal{N})$, where $V_\pi(\mathcal{N})$ is the vector space

$$V_\pi(\mathcal{N}) \ = \ \mathrm{span}\ \{\ \pi(n)v - v \mid n \in \mathcal{N} \text{ and } v \in V_\pi\ \}.$$

Let $\mathcal{N}'$ be the group of k-rational points of the unipotent radical N' of the parabolic k-subgroup P' containing M and opposite to P. A compact open subgroup $\mathcal{J}$ of $\mathcal{G}$ is said to have the Iwahori decomposition with respect to the parabolic $\mathcal{P}$ and the Levi factor $\mathcal{M}$ if

$$\mathcal{J} \ = \ (\mathcal{J} \cap \mathcal{N}') \cdot (\mathcal{J} \cap \mathcal{M}) \cdot (\mathcal{J} \cap \mathcal{N}).$$

The following fundamental theorem of Jacquet and Harish-Chandra (see [Cas:3.3.1] or [Si:2.3.6]) shows the importance of an Iwahori decomposition.

2.2. Theorem. *Let (π, V_π) be an admissible representation of $\mathcal{G}$. If $\mathcal{P} = \mathcal{M}\mathcal{N}$ is a parabolic subgroup of $\mathcal{G}$ and $\mathcal{J}$ is an open compact subgroup which has the Iwahori decomposition with respect to $\mathcal{P}$ and the Levi factor $\mathcal{M}$, then the map*

$$J_{\mathcal{N}} : V_\pi \longrightarrow J_{\mathcal{N}}(V_\pi)$$

yields a surjection

$$V_\pi^{\mathcal{J}} \longrightarrow (J_{\mathcal{N}}(V_\pi))^{\mathcal{J} \cap \mathcal{M}}.$$

In particular, as there is a sequence of open compact subgroups of $\mathcal{G}$ which admit Iwahori decomposition and which constitute a fundamental system of neighborhoods of the identity, the Jacquet module $J_{\mathcal{N}}(V_\pi)$ is an admissible representation of $\mathcal{M}$.

We note here that $J_{\mathcal{N}}(V_\pi)$ is a $\mathcal{M}$-module of finite length.

2.3. If $\mathrm{Ind}_{\mathcal{MN}}^{\mathcal{G}}$ is the unnormalized induction functor from $\mathcal{P} = \mathcal{MN}$ to $\mathcal{G}$, it is elementary (see [Cas:3.2.4] or [Car:II]) that $J_{\mathcal{N}}$ is the left adjoint of $\mathrm{Ind}_{\mathcal{MN}}^{\mathcal{G}}$ i.e. given an irreducible admissible representation (σ, V_σ) of $\mathcal{M}$, there is a canonical identification

$$\mathrm{Hom}_{\mathcal{G}}(V_\pi, \mathrm{Ind}_{\mathcal{MN}}^{\mathcal{G}} V_\sigma) \;=\; \mathrm{Hom}_{\mathcal{M}}(J_{\mathcal{N}}(V_\pi), V_\sigma).$$

2.4. Given a parabolic subgroup $\mathcal{P}$ and a Levi decomposition $\mathcal{P} = \mathcal{MN}$, a parabolic subgroup $\mathcal{Q}$ for which $\mathcal{M}$ is a Levi factor is called an $\mathcal{M}$-associate of $\mathcal{P}$. A representation σ of $\mathcal{M}$ can be inflated to any $\mathcal{M}$-associate $\mathcal{Q}$ of $\mathcal{P}$ and induced to $\mathcal{G}$.

2.5. Theorem. *Suppose σ is an irreducible absolutely cuspidal representation of $\mathcal{M}$ and π is an irreducible subquotient of the induced representation $Ind_{\mathcal{MN}}^{\mathcal{G}}\sigma$. Then, there exists an $\mathcal{M}$-associate $\mathcal{MU}$ (resp. $\mathcal{MV}$) such that π is a subrepresentation (resp. quotient) of the induced representation $Ind_{\mathcal{MU}}^{\mathcal{G}}\sigma$ (resp. $Ind_{\mathcal{MV}}^{\mathcal{G}}\sigma$). Furthermore, for any two $\mathcal{M}$-associate parabolic subgroups $\mathcal{P}_1 = \mathcal{MN}_1$ and $\mathcal{P}_2 = \mathcal{MN}_2$, the Jordan–Holder factors of $Ind_{\mathcal{MN}_1}^{\mathcal{G}}\sigma$ and $Ind_{\mathcal{MN}_2}^{\mathcal{G}}\sigma$ coincide.*

3. Unrefined minimal K-types for reductive groups

3.1. Let K be a fixed maximal unramified extension of k and $\Gamma = \mathrm{Gal}(K/k)$. Let $\mathfrak{o}$ and $\mathfrak{O}$ denote the ring of integers of k and K respectively; $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ and $\mathfrak{f}$ (resp. $\mathfrak{F}$) be the residue field of k (resp. K). Let $\mathcal{D}G$ be the derived group of G. The Bruhat–Tits buildings $\mathcal{B}(\mathcal{D}G, K)$ and $\mathcal{B}(\mathcal{D}G, k)$ of $\mathcal{D}G/K$ and $\mathcal{D}G/k$ respectively are canonically defined. The (*enlarged*) Bruhat–Tits building $\mathcal{B}(G, K)$ (resp. $\mathcal{B}(G, k)$) of G/K (resp. of G/k or of $\mathcal{G}$) is the product of $\mathcal{B}(\mathcal{D}G, K)$ (resp. $\mathcal{B}(\mathcal{D}G, k)$) with $X_*(C) \otimes_{\mathbb{Z}} \mathbb{R}$; where C is the maximal K-split (resp. k-split) torus contained in the center of G. There is a natural action of the Galois group Γ on $\mathcal{B}(G, K)$ and $\mathcal{B}(G, k)$ can (and will) be identified with the subset of points of $\mathcal{B}(G, K)$ fixed under Γ.

To each point x of the building $\mathcal{B}(G, k)$ of G/k, Bruhat and Tits [BT2] have associated a subgroup of $G(K)$, the *parahoric subgroup* determined by x, to be denoted by G_x here (we hope this notation will not cause any confusion), which is a certain subgroup of finite index in the isotropy group at x. (The parahoric subgroup is defined to be the inverse image in the isotropy at x of the identity component of reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated with the isotropy subgroup. If G is a semi-simple simply connected group, then the parahoric subgroup coincides with the isotropy subgroup.) The parahoric subgroup is "defined" over k i.e. it is stable under Γ; the subgroup $\mathcal{G} \cap G_x$ will be denoted by $\mathcal{G}_x$ and it is by definition the parahoric subgroup of $\mathcal{G}$ determined by (or associated with) the point x; it is a compact-open subgroup of $\mathcal{G}$. A minimal parahoric subgroup is called an *Iwahori subgroup*. It is known that the Iwahori subgroups are conjugate to each other under $\mathcal{G}$.

3.2. *Filtrations of parahoric subgroups.* Let S be a maximal k-split torus of G and T be a maximal K-split torus defined over k and containing S. Let Z be the centralizer of T in G. Then as G is quasi-split over K, Z is a torus and it is defined over k since T is. Let

L be the splitting field of Z/K; it is a totally ramified finite Galois extension of K. Let $\ell = [L : K]$ and ω be the additive valuation of L such that $\omega(L^\times) = \mathbb{Z}$.

Let $\Phi(\subset X^*(T))$ be the set of roots of G with respect to T. For $b \in \Phi$, let U_b be the corresponding root subgroup; it is a connected unipotent subgroup of G defined over K and normalized by Z; let G_b be the subgroup of G generated by U_b and U_{-b}. Each root $b \in \Phi$ determines a unique (up to K-isomorphism) extension L_b of K contained in L such that if b is nonmultipliable, then U_b, and if b is multipliable, then U_b/U_{2b}, is K-isomorphic to $R_{L_b/K}(\mathrm{Add})$. In case G/K is a product of a torus and certain absolutely quasi-simple connected K-groups, then the degree $[L_b : K] \leqq 3$, for all $b \in \Phi$, but in general these degrees can be arbitrary.

The apartment A of the Bruhat–Tits building of G/K associated with the torus T is an affine space under $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Let F denote the space of $\mathbb{R}$-valued affine-linear functions on A and let the constant function which at every point of A takes the value 1 be denoted by δ. The apartment $A(S)$ associated with the maximal k-split torus S in the building $\mathcal{B}(G,k)$ is $A \cap \mathcal{B}(G,k)$.

Let $\Psi(\subset F)$ be the set of affine roots of G relative to T, K (and the valuation ω on K). For $\psi \in \Psi$, let U_ψ be the subgroup defined exactly as in [MP: 2.4]. Let x be a point of $A(S)$ and G_x be the associated parahoric subgroup of $G(K)$. Then the subgroup $Z_0 := Z(K) \cap G_x$ is a subgroup of the maximal bounded subgroup of $Z(K)$ of finite index; it contains the maximal bounded subgroup of $T(K)$, and as the notation suggests, it does not depend on $x(\in A(S))$. For any positive integer n, let

$$Z_n = \{z \in Z_0 \mid \omega(\chi(z) - 1) \geqq n \text{ for all characters } \chi \text{ of } Z\}.$$

Now for a point x in the apartment $A(S)$ and $r \geqq 0$, let $G_{x,r}$ be the subgroup generated by the Z_n for $n \geqq r$, and the U_ψ for $\psi \in \Psi$ such that $\psi(x) \geqq r$. It is obvious that $G_{x,0} = G_x$ and if $s \geqq r$, then $G_{x,s} \subset G_{x,r}$. For $r \geqq 0$, let $G_{x,r^+} = \cup_{s>r} G_{x,s}$. It follows from [T: 1.4.2] that $G_{x,r}$ is a normal subgroup of G_x and in fact for $r, s \geqq 0$, the commutator subgroup $[G_{x,r}, G_{x,s}]$ is contained in $G_{x,r+s}$. Therefore, for $r > 0$, $[G_{x,0^+}, G_{x,r}] \subset G_{x,r^+}$ and so in particular, $G_{x,r}/G_{x,r^+}$ is abelian. The conjugation action of G_x on $G_{x,r}$ induces a natural action of the group $G_x/G_{x,0^+}$ on $G_{x,r}/G_{x,r^+}$. The group $G_x/G_{x,0^+}$ can be identified with the group of $\mathfrak{F}$-rational points of the maximal reductive quotient M_x of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated to the parahoric subgroup G_x.

Since G_x acts transitively on the set of apartments of the Bruhat–Tits building of G/K containing x ([BT2: Proposition 4.6.28(iii)]), and for $r \geqq 0$, $G_{x,r}$ is a normal subgroup of G_x, we conlcude that the filtration of G_x introduced above is independent of the choice of the apartment containing x.

For each $r \geqq 0$, the subgroup $G_{x,r}$ is stable under the Galois group Γ and we denote the subgroup $\mathcal{G}_x \cap G_{x,r}$ (resp. $\mathcal{G}_x \cap G_{x,r^+}$) by $\mathcal{G}_{x,r}$ (resp. $\mathcal{G}_{x,r^+}$). The subgroups $\mathcal{G}_{x,r}$ and $\mathcal{G}_{x,r^+}$ are open normal subgroups of the parahoric group $\mathcal{G}_x$. For $y = g \cdot x$, $g \in \mathcal{G} = G(k)$, set $G_{y,r} = gG_{x,r}g^{-1}$, $G_{y,r^+} = gG_{x,r^+}g^{-1}$; $\mathcal{G}_{y,r} = g\mathcal{G}_{x,r}g^{-1}$ and $\mathcal{G}_{y,r^+} = g\mathcal{G}_{x,r^+}g^{-1}$. These subgroups are well defined (i.e. they depend only on y and not on the choice of g). In the sequel we shall often denote the pro-nil radical $\mathcal{G}_{y,0^+}$ of $\mathcal{G}_y$ by $\mathcal{G}_y^+$.

3.3. *The associated filtrations of* $\mathfrak{g}$ *and its dual* $\mathfrak{g}^*$. The construction of the filtrations of $\mathfrak{g}$ and $\mathfrak{g}^*$ given in [MP:§3] can be imitated to get for $r \in \mathbb{R}$, and x in the Bruhat–Tits building of G/k, filtration lattices $\mathfrak{g}_{x,r}$ and $\mathfrak{g}^*_{x,-r}$ in the more general setup of this paper once we make the following definitions. Let L be as in 3.2 and for $b \in \Phi$, let L_b be as in 3.2. We fix a uniformizing element ϖ_b of L_b. As before, let $\ell = [L : K]$. For $b \in \Phi$, let $\ell_b = [L : L_b] = \ell/[L_b : K]$. For an affine root ψ whose derivative (or gradient) is b, we set $\ell_\psi = \ell_b$. We note that $\psi + t\delta$ is again an affine root if and only if t is an integral multiple of ℓ_ψ.

Let $\mathfrak{g}_{x,r^+} = \cup_{s>r}\mathfrak{g}_{x,s}$ (resp. $\mathfrak{g}^*_{x,-r^+} = \cup_{s<r}\mathfrak{g}^*_{x,-s}$). For every r, $\mathfrak{g}_{x,r}$ (resp. $\mathfrak{g}^*_{x,-r}$) is stable under the adjoint (resp. coadjoint) action of $\mathcal{G}_x$ and the induced action of the subgroup $\mathcal{G}_x^+$ on $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ (resp. $\mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,-r^+}$) is trivial. So there is an induced action of $\mathsf{M}_x(\mathfrak{f}) = \mathcal{G}_x/\mathcal{G}_x^+$ on $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ as well as on $\mathfrak{g}^*_{x,-r}/\mathfrak{g}^*_{x,-r^+}$. For $r > 0$, there is natural $\mathsf{M}_x(\mathfrak{f})$-equivariant isomorphism of $\mathcal{G}_{x,r}/\mathcal{G}_{x,r^+}$ onto $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$.

Given a point x of the Bruhat–Tits building of $\mathcal{G}$, there is a monotone increasing sequence $\{r_i \mid i \in \mathbf{N} \cup \{0\}\}$ of nonnegative real numbers such that $r_0 = 0$, for all i, $\mathcal{G}_{x,r_{i-1}} \neq \mathcal{G}_{x,r_i}$, $\mathfrak{g}_{x,r_{i-1}} \neq \mathfrak{g}_{x,r_i}$, and for $r_{i-1} < s \leqq r_i$, $\mathcal{G}_{x,s} = \mathcal{G}_{x,r_i}$, $\mathfrak{g}_{x,s} = \mathfrak{g}_{x,r_i}$ (cf. [MP: 3.4]). (Equivalently, $\mathfrak{g}^*_{x,-r_{i-1}} \neq \mathfrak{g}^*_{x,-r_i}$ and for $r_{i-1} \leqq s < r_i$, $\mathfrak{g}^*_{x,-s} = \mathfrak{g}^*_{x,-r_{i-1}}$.)

3.4. *Definition of nilpotence and unrefined minimal* K*-type.* We shall say that an element X of $\mathfrak{g}^*$ is *nilpotent* if there is a 1-parameter subgroup

$$\lambda : \quad GL_1 \longrightarrow G,$$

defined over k, such that $\operatorname*{Lim}_{t\to 0} \lambda(t)X = 0$. Note that in case k is of characteristic zero, according to a theorem of Kempf and Rousseau ([K],[R]), the last condition is equivalent to the condition that the Zariski-closure of the G-orbit $G \cdot X$ contains 0. Thus for local fields of characteristic zero, the above definition of nilpotence is equivalent to that given in [MP:3.5].

For $i > 0$, the Pontrjagin dual of $\mathcal{G}_{x,r_i}/\mathcal{G}_{x,r_{i+1}}$ can be identified (after a choice of an additive character of the prime field of $\mathfrak{f}$) with $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$ (cf. [MP: 3.7, 3.8]). A character χ of $\mathcal{G}_{x,r_i}/\mathcal{G}_{x,r_{i+1}}$ is said to be *nondegenerate* if the coset $X + \mathfrak{g}^*_{x,-r_{i-1}}$ corresponding to it does not contain any nilpotent elements. The nondegeneracy of a character does not depend on the choice of the additive character.

An *unrefined minimal* K*-type* is a pair $(\mathcal{G}_{x,r}, \chi)$, where $x \in \mathcal{B}(G,k)$, r is a nonnegative real number such that $\mathcal{G}_{x,r} \neq \mathcal{G}_{x,r^+}$, χ is a representation of $\mathcal{G}_{x,r}$ trivial on $\mathcal{G}_{x,r^+}$ and

(i) If $r = 0$, then χ is a cuspidal representation of the reductive group $\mathsf{M}_x(\mathfrak{f}) = \mathcal{G}_x/\mathcal{G}_x^+$ inflated to $\mathcal{G}_x$.
(ii) If $r > 0$, then χ is a nondegenerate character of $\mathcal{G}_{x,r}/\mathcal{G}_{x,r^+}$.

We define associativity of two unrefined minimal K-types as in [MP:5.1]: Two minimal K-types $(\mathcal{G}_{x,r}, \chi)$ and $(\mathcal{G}_{y,s}, \xi)$ are said to be *associates* if they have the same depth (i.e. $r = s$), and

(i) In case $r = 0$, there is an element $g \in \mathcal{G}$ such that $\mathcal{G}_x \cap \mathcal{G}_{gy}$ surjects onto both $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{gy}(\mathfrak{f})$ and χ is isomorphic to ξ^g.

(ii) In case $r > 0$, the $\mathcal{G}$-orbit of the coset which realizes χ intersects the coset which realizes ξ.

Given the above setup, Theorem 5.2 in [MP] generalizes (with the same proof) to

3.5. Theorem. *Given an irreducible admissible complex representation (π, V_π) of $\mathcal{G}$, there is a nonnegative rational number $\varrho(\pi)$ with the following properties.*

(1) *For some $x \in \mathcal{B}(G,k)$, the space $V_\pi^{\mathcal{G}_{x,\varrho(\pi)+}}$ of $\mathcal{G}_{x,\varrho(\pi)+}$-fixed vectors is nonzero and $\varrho(\pi)$ is the smallest number with this property.*

(2) *For any $y \in \mathcal{B}(G,k)$, if $W = V_\pi^{\mathcal{G}_{y,\varrho(\pi)+}} \neq \{0\}$, then*
   - *i) if $\varrho(\pi) = 0$, any irreducible $\mathcal{G}_{y,\varrho(\pi)}$-submodule of W contains an unrefined minimal* K*-type of depth zero of a parahoric $\mathcal{G}_z \subset \mathcal{G}_y$;*
   - *ii) if $\varrho(\pi) > 0$, any irreducible $\mathcal{G}_{y,\varrho(\pi)}$-submodule of W is an unrefined minimal* K*-type.*

Moreover, any two unrefined minimal K*-types contained in π are associates of each other.*

The rational number $\varrho(\pi)$ is by definition the *depth* of the irreducible representation π.

4. Depth and Jacquet functor

4.1. Let $P = MN$ and N' be as in §2 and $\mathcal{M}$, $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{P}$ be the group of k-rational points of M, N, N' and P respectively. Important examples of open compact subgroups which have Iwahori decomposition are given by the filtration subgroups introduced in §3. Let S be a maximal k-split torus of G contained in M. Then M contains the centralizer $Z_G(S)$ of S. Let $A(S)$ be the apartment in the building of $\mathcal{G}$ corresponding to S and for $x \in A(S)$, let $\mathcal{G}_x$ be the parahoric subgroup of $\mathcal{G}$ associated with x. For $r \geqq 0$, let $\mathcal{G}_{x,r}$ (resp. $\mathcal{G}_{x,r^+}$) be the filtration subgroup (of the parahoric subgroup $\mathcal{G}_x$) described in §3.2. In view of the description of $\mathcal{G}_{x,r}$ in terms of affine roots given in §3.2 and the results contained in §§6.4.9 and 6.4.48 of [BT1], we have the following result.

4.2. Theorem. *For $S \subset M$, any point $x \in A(S)$ and any $r > 0$, the filtration subgroup $\mathcal{G}_{x,r}$ of the parahoric subgroup $\mathcal{G}_x$ has the Iwahori decomposition with respect to $\mathcal{P}$ and the Levi factor $\mathcal{M}$.*

4.3. Let $L(M)$, $L(N)$ and $L(N')$ be the Lie algebras of M, N and N' respectively and let $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{n}'$ be the space of k-rational points of the respective Lie algebras. We have

$$\mathfrak{g} \;=\; \mathfrak{n}' \oplus \mathfrak{m} \oplus \mathfrak{n}.$$

The vector space duals of $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{n}'$ will be denoted by $\mathfrak{m}^*$, $\mathfrak{n}^*$ and ${\mathfrak{n}'}^*$ respectively. We shall view the dual $L(M)^*$ of $L(M)$(resp. $\mathfrak{m}^*$) as an M-module (resp. $\mathcal{M}$-module) under the coadjoint action of M (resp. $\mathcal{M}$).

There is a natural $\mathcal{M}$-module map from $\mathfrak{g}^*$ to $\mathfrak{m}^*$ given by restriction. For $X \in \mathfrak{g}^*$, we shall denote its restriction to $\mathfrak{m}$ by $X_{\mathfrak{m}}$; for a subset Ξ of $\mathfrak{g}^*$, we shall let $\Xi_{\mathfrak{m}}$ denote the subset $\{X_{\mathfrak{m}} \mid X \in \Xi\}$ of $\mathfrak{m}^*$.

The dual $\mathfrak{m}^*$ will be identified with the $\mathcal{M}$-submodule:

$$\{\ X \in \mathfrak{g}^* \mid X|_{\mathfrak{n}'} = 0 \text{ and } X|_{\mathfrak{n}} = 0\ \};$$

$\mathfrak{n}^*$ and $\mathfrak{n}'^*$ have identifications with similarly defined $\mathcal{M}$-submodules of $\mathfrak{g}^*$. With these identifications, we have

$$\mathfrak{g}^* \;=\; \mathfrak{n}'^* \oplus \mathfrak{m}^* \oplus \mathfrak{n}^*.$$

For $x \in A(S)$ and $r \in \mathbb{R}$, let $\mathfrak{g}^*_{x,-r}$ be as in 3.3 and let $\mathfrak{m}^*_{x,-r}$, $\mathfrak{n}^*_{x,-r}$ and $\mathfrak{n}'^*_{x,-r}$ denote $\mathfrak{m}^* \cap \mathfrak{g}^*_{x,-r}$, $\mathfrak{n}^* \cap \mathfrak{g}^*_{x,-r}$ and $\mathfrak{n}'^* \cap \mathfrak{g}^*_{x,-r}$ respectively. We have

$$\mathfrak{g}^*_{x,-r} \;=\; \mathfrak{n}'^*_{x,-r} \oplus \mathfrak{m}^*_{x,-r} \oplus \mathfrak{n}^*_{x,-r}. \tag{4.3.1}$$

Let $\mathcal{M}_x = \mathcal{M} \cap \mathcal{G}_x$, $\mathcal{N}_x = \mathcal{N} \cap \mathcal{G}_x$ and $\mathcal{N}'_x = \mathcal{N}' \cap \mathcal{G}_x$. For $r \geqq 0$, let $\mathcal{M}_{x,r} = \mathcal{M} \cap \mathcal{G}_{x,r}$, $\mathcal{N}_{x,r} = \mathcal{N} \cap \mathcal{G}_{x,r}$ and $\mathcal{N}'_{x,r} = \mathcal{N}' \cap \mathcal{G}_{x,r}$ (resp. $\mathcal{M}_{x,r^+} = \mathcal{M} \cap \mathcal{G}_{x,r^+}$, $\mathcal{N}_{x,r^+} = \mathcal{N} \cap \mathcal{G}_{x,r^+}$ and $\mathcal{N}'_{x,r^+} = \mathcal{N}' \cap \mathcal{G}_{x,r^+}$). We shall denote $\mathcal{M}_{x,0^+}$ also by $\mathcal{M}_x^+$. Then since M contains $Z_G(S)$, and so in particular it contains the torus Z, $\mathcal{M}_x$ is the parahoric subgroup of $\mathcal{M}$ associated with the point x (note that the apartment $A(S)$ is also the apartment corresponding to the maximal k-split torus S in the building of $\mathcal{M}$). The subgroup $\mathcal{M}_{x,r}$ of $\mathcal{M}_x$ coincides with the filtration subgroup, and $\mathfrak{m}^*_{x,r}$ coincides with the filtration lattice in $\mathfrak{m}^*$, associated with x and r, obtained by using the construction given in §3 for the reductive group M in place of G.

Let $\{r_i\}$ be the monotone increasing sequence of nonnegative real numbers associated to x in 3.3. Then the Pontrjagin dual, for $i \geqq 1$, of the abelian group $\mathcal{G}_{x,r_i}/\mathcal{G}_{x,r_{i+1}}$ has an identification with $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$ (see 3.3). This provides an identification of the Pontrjagin dual of $\mathcal{M}_{x,r_i}/\mathcal{M}_{x,r_{i+1}}$ with $\mathfrak{m}^*_{x,-r_i}/\mathfrak{m}^*_{x,-r_{i-1}}$.

4.4. Suppose (π, V_π) is an irreducible admissible representation of $\mathcal{G}$ such that $J_{\mathcal{N}}(V_\pi) \neq \{0\}$. Let (σ, V_σ) be an irreducible subquotient of the $\mathcal{M}$-module $J_{\mathcal{N}}(V_\pi)$ and $r := \varrho(\sigma)$ be its depth. Choose a point $x \in A(S)$ so that

(1) $V_\sigma^{\mathcal{M}_{x,r^+}} \neq \{0\}$
(2) the action of $\mathcal{M}_{x,r}$ on $V_\sigma^{\mathcal{M}_{x,r^+}}$ contains an unrefined minimal K-type.

By Theorem 2.2, $V_\pi^{\mathcal{G}_{x,r^+}} \neq \{0\}$.

4.5. Theorem. *The $\mathcal{G}_{x,r}/\mathcal{G}_{x,r^+}$-constituents of $V_\pi^{\mathcal{G}_{x,r^+}}$ contain unrefined minimal K-types. (In particular $\varrho(\sigma) = \varrho(\pi)$ for any irreducible subquotient σ of $J_{\mathcal{N}}(V_\pi)$.) Moreover, if $r > 0$ and $(\mathcal{M}_{x,r}, \chi)$ is an unrefined minimal K-type occuring in $V_\sigma^{\mathcal{M}_{x,r^+}}$, then $V_\pi^{\mathcal{G}_{x,r^+}}$ contains an unrefined minimal K-type of the form $(\mathcal{G}_{x,r}, \xi)$ such that the restriction of ξ*

to $\mathcal{M}_{x,r}$ is χ and its restriction to $\mathcal{N}_{x,r}$ and $\mathcal{N}'_{x,r}$ are trivial; consequently the coset Ξ corresponding to ξ contains an element of $\mathfrak{m}^*$.

The assertion is clear when $r = 0$; any $\mathcal{G}_x/\mathcal{G}_x^+$-irreducible constituent of $V_\pi^{\mathcal{G}_x^+}$ contains a cuspidal representation of a parabolic subgroup of the reductive group $\mathsf{M}_x(\mathfrak{f}) = \mathcal{G}_x/\mathcal{G}_x^+$. We shall make a more precise statement in this, the depth zero, setting in §6. We can therefore now focus on the case $r > 0$ in which case $\mathcal{M}_{x,r}/\mathcal{M}_{x,r^+}$ and $\mathcal{G}_{x,r}/\mathcal{G}_{x,r^+}$ are both abelian. The proof in this case is based on a property of the characters of the abelian group $\mathcal{G}_{x,r_i}/\mathcal{G}_{x,r_{i+1}}$ $(i > 0)$ which we shall now formulate (4.7). We begin with a lemma.

4.6. Lemma. *Let H be a connected reductive group defined over a field F; Q be a parabolic F-subgroup of H and let $\rho : H \longrightarrow GL(V)$ be a finite dimensional F-rational representation of H on a F-vector space V. Let $v \in V$ and $\lambda : GL_1 \longrightarrow H$ be a 1-parameter subgroup defined over F such that $\mathop{Lim}\limits_{t\to 0} \rho(\lambda(t))v = 0$. Then there exists a 1-parameter subgroup $\mu : GL_1 \longrightarrow Q$ defined over F such that $\mathop{Lim}\limits_{t\to 0} \rho(\mu(t))v = 0$.*

Proof. The group $\mathcal{Q}_\lambda = \{x \in H(F) \mid \mathop{\mathrm{Lim}}\limits_{t\to 0} \lambda(t)x\lambda(t)^{-1} \text{ exists}\}$ is the group of F-rational points of a parabolic subgroup Q_λ of H defined over F. According to Proposition 20.7(i) of [B2], $Q_\lambda \cap Q$ contains a maximal F-split torus T. Also, λ is contained in a maximal F-split torus T' of Q_λ. Since both T and T' are maximal F-split tori of Q_λ, there exists a $q \in \mathcal{Q}_\lambda$ such that $qT'q^{-1} = T$. Let $\mu(t) = q\lambda(t)q^{-1}$. Then $\mu \subset T \subset Q_\lambda \cap Q \subset Q$ and $\mathop{\mathrm{Lim}}\limits_{t\to 0} \rho(\mu(t))v = 0$. □

4.7. Proposition. *Let $\Xi = X + \mathfrak{g}^*_{x,-r_{i-1}}$ be a coset which contains a nilpotent element, then there exists an element $p \in \mathcal{M}_x\mathcal{N}_x$ so that the coset $(p\Xi)_\mathfrak{m}$ also contains a nilpotent element.*

Proof. We assume (as we may) that $X \in \Xi$ is nilpotent. Let λ be a 1-parameter subgroup of G defined over k such that $\mathop{\mathrm{Lim}}\limits_{t\to 0} \lambda(t)X = 0$.

Let M_x be the quotient of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated to the parahoric subgroup $\mathcal{G}_x$ by its unipotent radical. M_x is a connected reductive $\mathfrak{f}$-group and the maximal k-split torus S of G gives rise to a maximal $\mathfrak{f}$-split torus S of M_x. According to Proposition 4.3 of [MP], the M_x-orbit through the image $\overline{X}$ of X in $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$ contains zero in its Zariski-closure. By a theorem of Kempf and Rousseau ([K], [R]), there is a 1-parameter subgroup $\overline{\lambda}$ of M_x, defined over $\mathfrak{f}$, such that $\mathop{\mathrm{Lim}}\limits_{t\to 0} \overline{\lambda}(t)\overline{X} = 0$. Now since the image of $\mathcal{M}_x\mathcal{N}_x$ in M_x is the group of $\mathfrak{f}$-rational points of a parabolic $\mathfrak{f}$-subgroup P, and P clearly contains the maximal $\mathfrak{f}$-split torus S, we conclude, using the preceding lemma and conjugacy of maximal $\mathfrak{f}$-split tori of P under $\mathsf{P}(\mathfrak{f})$, that there is an element $p \in \mathcal{M}_x\mathcal{N}_x$ and a 1-parameter subgroup $\overline{\mu}$ contained in S such that $\mathop{\mathrm{Lim}}\limits_{t\to 0} \overline{p}^{-1}\overline{\mu}(t)\overline{p}\overline{X} = 0$; where $\overline{p}$ is the image of p in $\mathsf{M}_x(\mathfrak{f})$. Let μ be the lift of the 1-parameter subgroup $\overline{\mu}$ of S to S. Then clearly, the limit, as $t \to 0$, of the image of $\mu(t)pX$ in $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$ is zero. Now for a *positive* integer n, let

$$V_n = \{Z \in \mathfrak{g}^*_{x,-r_i} \mid \mu(t)Z = t^n Z\},$$

and

$$\overline{V}_n = \{\overline{Z} \in \mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}} \mid \overline{\mu}(t)\overline{Z} = t^n\overline{Z}\}.$$

Let V (resp. $\overline{V}$) be the submodule spanned by the V_n's (resp. $\overline{V}_n$'s). Then for each n, V_n projects onto $\overline{V}_n$ under the natural projection $\mathfrak{g}^*_{x,-r_i} \longrightarrow \mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$ and so V projects onto $\overline{V}$. It is clear that for $Z \in \mathfrak{g}^*_{x,-r_i}$ (resp. $\overline{Z} \in \mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$), $\operatorname*{Lim}_{t\to 0} \mu(t)Z = 0$ (resp. $\operatorname*{Lim}_{t\to 0} \overline{\mu}(t)\overline{Z} = 0$) if and only if $Z \in V$ (resp. $\overline{Z} \in \overline{V}$). As $\operatorname*{Lim}_{t\to 0} \overline{p}^{-1}\overline{\mu}(t)\overline{p}\overline{X} = 0$, $\overline{p}\overline{X} \in \overline{V}$ and hence, there is a $Y \in V$ which projects onto $\overline{p}\overline{X}$. Such a Y lies in $p\Xi$ and $\operatorname*{Lim}_{t\to 0} \mu(t)Y = 0$. Now since

$$\mathfrak{g}^*_{x,-r_i} = \mathfrak{n}'^*_{x,-r_i} \oplus \mathfrak{m}^*_{x,-r_i} \oplus \mathfrak{n}^*_{x,-r_i},$$

and each of the $\mathfrak{n}'^*$, $\mathfrak{m}^*$ and $\mathfrak{n}^*$ is stable under $\mu(\subset S)$, we conclude that $\operatorname*{Lim}_{t\to 0} \mu(t)Y_{\mathfrak{m}} = 0$. So $Y_{\mathfrak{m}}(\in (p\Xi)_{\mathfrak{m}})$ is nilpotent. This proves the proposition. □

Proof of Theorem 4.5. We fix a $\mathcal{M}$-submodule $\mathcal{V}_\sigma$ of $J_{\mathcal{N}}(V_\pi)$ which projects onto V_σ. As stated above, we can assume that $r = \varrho(\sigma) > 0$. Choose a nonzero $\overline{v} \in V_\sigma^{\mathcal{M}_{x,r^+}}$ which transforms under $\mathcal{M}_{x,r}/\mathcal{M}_{x,r^+}$ by homotheties according to the nondegenerate character χ. Let $\mathcal{C} = \mathcal{N}'_{x,r}\mathcal{M}_{x,r^+}\mathcal{N}_{x,r}$; $\mathcal{C}$ is a compact-open subgroup of $\mathcal{G}$, normal in $\mathcal{G}_{x,r}$, and $\mathcal{G}_{x,r}/\mathcal{C}$ ($\cong \mathcal{M}_{x,r}/\mathcal{M}_{x,r^+}$) is an abelian group. It follows from Theorem 2.2 that there is a $v \in V_\pi^{\mathcal{C}}$ such that $J_{\mathcal{N}}(v)$ lies in $\mathcal{V}_\sigma$ and its image in V_σ is $\overline{v}$. We can choose a vector w in the $\mathcal{M}_{x,r}$-submodule generated by v such that

(1) $J_{\mathcal{N}}(w) \neq 0$.
(2) w transforms under $\mathcal{G}_{x,r}$ according to a character ξ of $\mathcal{G}_{x,r}/\mathcal{G}_{x,r^+}$ whose restriction to $\mathcal{M}_{x,r}/\mathcal{M}_{x,r^+}$ is χ.

Then the restriction of ξ to $\mathcal{N}_{x,r}$ and $\mathcal{N}'_{x,r}$ are trivial, and hence the coset Ξ of $\mathfrak{g}^*_{x,-r_{i-1}}$ in $\mathfrak{g}^*_{x,-r_i}$ corresponding to ξ contains an element of $\mathfrak{m}^*$; here i is the positive integer such that $r = r_i$.

We claim that the coset Ξ does not contain any nilpotent elements. Suppose to the contrary that Ξ does contain a nilpotent element. According to Proposition 4.7, there exists then an element $p \in \mathcal{M}_x\mathcal{N}_x$ so that $(p\Xi)_{\mathfrak{m}}$ contains a nilpotent element. Consider the vector $w_p = \pi(p)w$. It lies in $V_\pi^{\mathcal{G}_{x,r^+}}$. Also, since $p \in \mathcal{P}$, and $J_{\mathcal{N}}$ is a $\mathcal{P}$-module homomorphism, the image $\overline{w_p}$ of w_p in V_σ is nonzero and $\mathcal{M}_{x,r}$ acts on it through the character corresponding to the coset $(p\Xi)_{\mathfrak{m}}$ which contains a nilpotent element. But as the depth of the representation σ is r, the coset $(p\Xi)_{\mathfrak{m}}$ can not contain any nilpotent elements. We conclude from this that Ξ can not contain any nilpotent elements either and hence ξ is an unrefined minimal K-type for π. □

4.8. An element of $\mathfrak{g}^*$ is said to be *semi-simple* if its G-orbit is closed in $L(G)^*$ in Zariski topology. We shall say that a semi-simple element Y of $\mathfrak{g}^*$ is *anisotropic* if the only k-split tori in G which fix Y are the k-split tori contained in the center of G. Since there is a nontrivial k-split torus contained in the center of M which is not central in G, and any such torus fixes every element of $\mathfrak{m}^*$, no element of $\mathfrak{m}^*$ is anisotropic. Given $i > 0$, a coset $\Xi = X + \mathfrak{g}^*_{x,-r_{i-1}}$ is said to be *anisotropic* if each $Y \in \Xi$ is anisotropic. If Ξ is an anisotropic coset, then it can't contain any element of $\mathfrak{m}^*$. A character ξ of $\mathcal{G}_{x,r_i}/\mathcal{G}_{x,r_{i+1}}$ will be called *anisotropic* if the corresponding coset Ξ is anisotropic.

As an immediate consequence of Theorem 4.5, we have

4.9. Corollary. *If π is an irreducible admissible representation which contains an anisotropic unrefined minimal* K*-type, then π is an absolutely cuspidal representation.*

This fact had been observed before in many special cases.

5. Parabolic induction and unrefined minimal K-types

In this section, we state and prove one of our main results which is the assertion that the depth of a representation behaves well under parabolic induction. As a preliminary to the main statement, we prove a lemma.

5.1. Lemma. *Suppose $P = MN$ is a parabolic k-subgroup of G and S is a maximal k-split torus of M. Then given $x \in A(S)$, there is a $y \in A(S)$ such that*

(1) *$\mathcal{M}_x = \mathcal{M}_y$ and*
(2) *$\mathcal{M}_y/\mathcal{M}_y^+ \cong \mathcal{G}_y/\mathcal{G}_y^+$.*

Proof. Let C be the maximal k-split torus contained in the center of M. Note that S contains C and M is precisely the centralizer of C in G. Therefore, the restriction to C of a k-root of G (relative to S) is trivial if and only if it is a k-root of M. It is obvious from this that there are points y of $A(S)$ lying in the set of translates of x by elements of $X_*(C) \otimes_{\mathbb{Z}} \mathbb{R}$ ($\subset X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$) such that if ψ is any affine root of G (relative to S and k) whose derivative (or gradient) is not a k-root of M, then $\psi(y) \neq 0$, whereas if the derivative of ψ is a k-root of M, then $\psi(x) = \psi(y)$. For any such y, clearly $\mathcal{M}_{y,r} = \mathcal{M}_{x,r}$ for all $r \geqq 0$, and the natural map $\mathcal{M}_y/\mathcal{M}_y^+ \to \mathcal{G}_y/\mathcal{G}_y^+$ is an isomorphism. $\square$

We now state the main result on depth and parabolic induction.

5.2. Theorem. *Suppose (σ, V_σ) is an admissible irreducible representation of $\mathcal{M}$ and (π, V_π) is an irreducible subquotient of the induced representation $Ind^{\mathcal{G}}_{\mathcal{M}\mathcal{N}}\sigma$. Then*

(1) *$\varrho(\pi) = \varrho(\sigma)$*
(2) *If σ is of depth zero, $(\mathcal{M}_x, \chi)$ is an unrefined minimal* K*-type of σ, and $y \in A(S)$ has the properties (1) and (2) of Lemma 5.1, then the unrefined minimal* K*-type χ (lifted to $\mathcal{G}_y/\mathcal{G}_y^+$ via property (2) of y) occurs in π.*
(3) *If $r = \varrho(\sigma) > 0$, given an unrefined minimal* K*-type $(\mathcal{M}_{x,r}, \chi)$ occuring in σ ($x \in A(S)$), there exists an unrefined minimal* K*-type $(\mathcal{G}_{x,r}, \xi)$ occuring in π such that the restriction of ξ to $\mathcal{M}_{x,r}$ is χ and its restriction to $\mathcal{N}_{x,r}$ and $\mathcal{N}'_{x,r}$ are trivial, so the coset Ξ corresponding to ξ contains an element of $\mathfrak{m}^*$.*

Proof. By Theorem 2.5, after replacing the inducing parabolic $\mathcal{P}$ by a suitable $\mathcal{M}$-associate, we may (and we shall) assume that π is an irreducible subrepresentation of $\mathrm{Ind}_{\mathcal{M}\mathcal{N}}^{\mathcal{G}} V_\sigma$. Then by Frobenius reciprocity (2.3), V_σ is an irreducible quotient of the Jacquet module $J_{\mathcal{N}}(V_\pi)$ and by Theorem 4.5, $\varrho(\pi) = \varrho(\sigma)$.

Let $\iota : J_{\mathcal{N}}(V_\pi) \to V_\sigma$ be a surjective $\mathcal{M}$-module homomorphism. Let $(\mathcal{M}_{x,r}, \chi)$ be an unrefined minimal K-type of σ; where $r = \varrho(\sigma)$. Now to prove (2) we note that according to Theorem 2.2, $\iota \circ J_{\mathcal{N}}$ maps $V_\pi^{\mathcal{G}_y^+}$ onto $V_\sigma^{\mathcal{M}_y^+}$. Since $\mathcal{G}_y/\mathcal{G}_y^+ \cong \mathcal{M}_y/\mathcal{M}_y^+$, the representation χ must in fact appear in $V_\pi^{\mathcal{G}_y^+}$. Part (3) is an immediate consequence of Theorem 4.5. $\square$

The analogue of part (1) of Theorem 5.2 for the Jacquet functor $J_{\mathcal{N}}$ is stated in Theorem 4.5. As analogues of part (2) and (3) of Theorem 5.2, we have the following result.

5.3. Proposition. *Let $\mathcal{P} = \mathcal{M}\mathcal{N}$ be a parabolic subgroup of $\mathcal{G}$ and π be an irreducible admissible representation of $\mathcal{G}$. Then*

(1) *If $\rho(\pi) = 0$ and $(\mathcal{M}_x, \chi)$ and $(\mathcal{M}_{x'}, \chi')$ are unrefined minimal K-types of depth zero in $J_{\mathcal{N}}(\pi)$, then there exist points y and y' so that $\mathcal{M}_y = \mathcal{M}_x$, $\mathcal{M}_{y'} = \mathcal{M}_{x'}$ and $\mathcal{G}_y/\mathcal{G}_y^+ \cong \mathcal{M}_y/\mathcal{M}_y^+$, $\mathcal{G}_{y'}/\mathcal{G}_{y'}^+ \cong \mathcal{M}_{y'}/\mathcal{M}_{y'}^+$ and the minimal K-types $(\mathcal{G}_y, \chi)$ and $(\mathcal{G}_{y'}, \chi')$ of π are associates.*

(2) *If $\varrho(\pi) > 0$, let $(\mathcal{M}_{x,r}, \chi)$ and $(\mathcal{M}_{y,r}, \chi')$ be two unrefined minimal K-types in $J_{\mathcal{N}}(\pi)$, and let $(\mathcal{G}_{x,r}, \xi)$ and $(\mathcal{G}_{y,r}, \xi')$ be unrefined minimal K-types provided by Theorem 4.5. Then $(\mathcal{G}_{x,r}, \xi)$ and $(\mathcal{G}_{y,r}, \xi')$ are associates.*

Proof. In part (1), the existence of the points y and y' follow from Lemma 5.1. The associativity statements in parts (1) and (2) is just associativity of unrefined minimal K-types in π. $\square$

6. Depth zero representations

In this section we concern ourselves only with depth zero representations. We call an unrefined minimal K-type of depth zero a minimal K-type. We formulate and prove for general depth zero representations analogues of Borel's results for representations with an Iwahori fixed vector. We begin with a preliminary result.

6.1. Proposition. *Let G be a connected reductive group defined over the finite field $\mathfrak{f}$ and let $\mathsf{P} = \mathsf{M}\mathsf{N}$ and $\mathsf{P}' = \mathsf{M}\mathsf{N}'$ be two associate parabolic $\mathfrak{f}$-subgroups of G, i.e. parabolic $\mathfrak{f}$-subgroups which share a Levi subgroup.*

(1) *If τ is a representation of $\mathsf{M}(\mathfrak{f})$, the two induced representations $\mathrm{Ind}_{\mathsf{M}(\mathfrak{f})\mathsf{N}(\mathfrak{f})}^{\mathsf{G}(\mathfrak{f})}\tau$ and $\mathrm{Ind}_{\mathsf{M}(\mathfrak{f})\mathsf{N}'(\mathfrak{f})}^{\mathsf{G}(\mathfrak{f})}\tau$ are equivalent.*

(2) *If (σ, V_σ) is a finite-dimensional complex representation of $\mathsf{G}(\mathfrak{f})$, then integration along $\mathsf{N}(\mathfrak{f})$, i.e. $A(v) = \int_{\mathsf{N}(\mathfrak{f})} \sigma(n)v\, dn$ defines an isomorphism of $V_\sigma^{\mathsf{N}'(\mathfrak{f})}$ with $V_\sigma^{\mathsf{N}(\mathfrak{f})}$.*

Proof. We begin with the proof of part (2). Let $e_{\mathsf{N}(\mathfrak{f})}$ and $e_{\mathsf{N}'(\mathfrak{f})}$ be the idempotent elements in the group algebra $\mathbb{C}[\mathsf{G}(\mathfrak{f})] = \mathcal{H}(\mathsf{G}(\mathfrak{f})//\{1\})$ determined by the subgroups $\mathsf{N}(\mathfrak{f})$ and $\mathsf{N}'(\mathfrak{f})$

respectively. Then given any representation (σ, V_σ), $V_\sigma^{\mathsf{N}(\mathfrak{f})} = \sigma(e_{\mathsf{N}(\mathfrak{f})})V_\sigma$ and $V_\sigma^{\mathsf{N}'(\mathfrak{f})} = \sigma(e_{\mathsf{N}'(\mathfrak{f})})V_\sigma$. The assertion in (2) that A is an isomorphism is equivalent to the statement that there exists an element $\xi \in e_{\mathsf{N}'(\mathfrak{f})}\mathbb{C}[\mathsf{G}(\mathfrak{f})]e_{\mathsf{N}(\mathfrak{f})}$ such that

$$\xi e_{\mathsf{N}(\mathfrak{f})}e_{\mathsf{N}'(\mathfrak{f})} = e_{\mathsf{N}'(\mathfrak{f})} \quad \text{and} \quad e_{\mathsf{N}(\mathfrak{f})}e_{\mathsf{N}'(\mathfrak{f})}\xi = e_{\mathsf{N}(\mathfrak{f})}.$$

Such a ξ is provided by Theorem 2.5 of [HL].

Part (1) is a consequence of Harish-Chandra's theory of cusp forms on $\mathsf{G}(\mathfrak{f})$ (see [HC]). As noted in [HL], it also follows immediately from Theorem 1.1 in [HL]. $\square$

The next result is the converse of the last part of Theorem 5.2 in [MP] which asserts that two unrefined minimal K-types $(\mathcal{G}_x, \sigma)$ and $(\mathcal{G}_y, \tau)$ which both occur in an irreducible admissible representation (π, V_π) of $\mathcal{G}$ are associates of one another.

6.2. Proposition. *Suppose (π, V_π) is an irreducible admissible representation of $\mathcal{G}$ of depth zero. If $(\mathcal{G}_x, \sigma)$ is a depth zero minimal K-type occuring in V_π, and $(\mathcal{G}_y, \tau)$ is an associate of $(\mathcal{G}_x, \sigma)$, then $(\mathcal{G}_y, \tau)$ also occurs in V_π.*

Proof. For any z in the Bruhat–Tits building of $\mathcal{G}$, let M_z be the quotient of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated to the parahoric subgroup $\mathcal{G}_z$ by its unipotent radical. Then $\mathcal{G}_z/\mathcal{G}_z^+ \cong \mathsf{M}_z(\mathfrak{f})$. So a representation of $\mathcal{G}_z$ which is trivial on $\mathcal{G}_z^+$ gives rise to a representation of $\mathsf{M}_z(\mathfrak{f})$, and conversely a representation of $\mathsf{M}_z(\mathfrak{f})$ inflates to a representation of $\mathcal{G}_z$ which is trivial on $\mathcal{G}_z^+$. In the sequel we shall not distinguish between a representation of $\mathcal{G}_z$ which is trivial on $\mathcal{G}_z^+$ and the corresponding representation of $\mathsf{M}_z(\mathfrak{f})$ and shall use the same symbol to denote the two representations. As $(\mathcal{G}_x, \sigma)$ and $(\mathcal{G}_y, \tau)$ are minimal K-types of depth zero, σ and τ are cuspidal representations of $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_y(\mathfrak{f})$ respectively.

After replacing the pair $(\mathcal{G}_y, \tau)$ by a suitable conjugate, we assume that the natural maps $\iota_x : \mathcal{G}_x \cap \mathcal{G}_y \to \mathsf{M}_x(\mathfrak{f})$ and $\iota_y : \mathcal{G}_x \cap \mathcal{G}_y \to \mathsf{M}_y(\mathfrak{f})$ are onto and let S be a maximal k-split torus of G such that the apartment $A(S)$ corresponding to S in the Bruhat–Tits building of $\mathcal{G}$ contains both x and y. Then an affine root ψ (of G relative to S and k) which vanishes at the point x must also vanish at the point y and vice versa (in particular, M_y has a natural identification with M_x as an algebraic $\mathfrak{f}$-group and with this identification, $\tau = \sigma$). From this we conclude that both x and y lie in an affine subspace B (of the apartment $A(S)$) which is given as the zero locus of a certain set of affine roots. The intersection of the affine subspace B with the vanishing hyperplane H of an affine root is either (i) empty, or (ii) equals B, or (iii) is an affine subspace of B of codimension 1. Let U be the open set (of B) which is the complement in B of the union of those H's which satisfy property (iii). The points x and y lie in U and U is a disjoint union of its (countably many) connected components. Given a point z in U, let $U(z)$ denote the connected component of U which contains z.

If z and z' are two arbitrary points in U, then the natural maps $\mathcal{G}_z \cap \mathcal{G}_{z'} \to \mathsf{M}_z(\mathfrak{f})$ and $\mathcal{G}_z \cap \mathcal{G}_{z'} \to \mathsf{M}_{z'}(\mathfrak{f})$ are surjective and the group $\mathsf{M}_{z'}$ has a natural identification with M_z as an algebraic $\mathfrak{f}$-group. We shall use the same symbol to denote the representations of $\mathcal{G}_z$

and $\mathcal{G}_{z'}$ trivial on $\mathcal{G}_z^+$ and $\mathcal{G}_{z'}^+$ respectively which correspond to each other in terms of the identification of $\mathsf{M}_z(\mathfrak{f})(=\mathcal{G}_z/\mathcal{G}_z^+)$ with $\mathsf{M}_{z'}(\mathfrak{f})(=\mathcal{G}_{z'}/\mathcal{G}_{z'}^+)$.

We say that two points z and z' are in adjacent components if the intersection $\overline{U(z)}\cap\overline{U(z')}$ of the closures of $U(z)$ and $U(z')$ in B is nonempty. Suppose z and z' are in adjacent components. We claim that an irreducible admissible representation (π,V_π) contains the minimal K-type $(\mathcal{G}_z,\sigma)$ of depth zero if and only if it contains the minimal K-type $(\mathcal{G}_{z'},\sigma)$. Clearly, we need only show that if (π,V_π) contains $(\mathcal{G}_z,\sigma)$, then it must contain $(\mathcal{G}_{z'},\sigma)$. Let v be a point of B which lies in the intersection $\overline{U(z)}\cap\overline{U(z')}$. The parahoric $\mathcal{G}_v$ contains both $\mathcal{G}_z$ and $\mathcal{G}_{z'}$ and the images of $\mathcal{G}_z$ and $\mathcal{G}_{z'}$ in M_v are the group of $\mathfrak{f}$-rational points of associate parabolic $\mathfrak{f}$-subgroups P_z and $\mathsf{P}_{z'}$ respectively of M_v. The quotient of P_z (resp. $\mathsf{P}_{z'}$) by its unipotent radical is $\mathfrak{f}$-isomorphic to M_z (resp. $\mathsf{M}_{z'}$). If V_π contains $(\mathcal{G}_z,\sigma)$, $V_\pi^{\mathcal{G}_v^+}$ must contain an irreducible representation ξ of $\mathcal{G}_v$ which is equivalent to a subrepresentation of the inflation to $\mathcal{G}_v$ of the induced representation $\mathrm{Ind}_{\mathsf{P}_z(\mathfrak{f})}^{\mathsf{M}_v(\mathfrak{f})}\sigma$. However, by Proposition 6.1(1) and the Frobenius reciprocity, any irreducible constituent of $\mathrm{Ind}_{\mathsf{P}_z(\mathfrak{f})}^{\mathsf{M}_v(\mathfrak{f})}\sigma$ contains the representation $(\mathsf{P}_{z'}(\mathfrak{f}),\sigma)$. Hence, V_π contains the minimal K-type $(\mathcal{G}_{z'},\sigma)$. This proves the claim. To complete the proof of Proposition 6.2 we note that for our original x and y, we can find a sequence of points $x=z_0,\ z_1,\ \dots\ ,z_n=y$ such that each z_i lies in U and $U(z_i)$ and $U(z_{i+1})$ are adjacent. It follows that if (π,V_π) contains $(\mathcal{G}_x,\sigma)$, then it also contains $(\mathcal{G}_y,\sigma)$. $\square$

6.3. We attach a Levi subgroup M to $\mathcal{G}_x$ as follows: Let S be a maximal k-split torus of G such that the apartment $A(S)$ corresponding to S contains x. Then S gives rise to a maximal $\mathfrak{f}$-split torus S of M_x. Let C be the maximal $\mathfrak{f}$-split torus contained in the center of M_x. Note that $\mathsf{C}\subset\mathsf{S}$. Lift C to S to get a subtorus C (of S). Let M be the centralizer of C in G and $\mathcal{M}=M(k)$. Then M is a Levi subgroup ([B2: 20.4]) and since the centralizer $Z_G(S)$ of S in G is contained in M, $\mathcal{M}_x=\mathcal{M}\cap\mathcal{G}_x$ is a parahoric subgroup of $\mathcal{M}$. *Note that C is contained in the center of G, or, equivalently, $M=G$, if and only if $\mathcal{G}_x$ is a maximal parahoric subgroup* (cf. [T: 3.5]).

6.4. Proposition.

(1) *$\mathcal{M}_x$ is a maximal parahoric subgroup of $\mathcal{M}$.*
(2) $\mathcal{M}_x/\mathcal{M}_x^+\ \cong\ \mathcal{G}_x/\mathcal{G}_x^+$ $(\cong\mathsf{M}_x(\mathfrak{f}))$.

Proof. Recall that there is a natural identification of the character groups $X^*(S)$ and $X^*(\mathsf{S})$ of S and S respectively. Since M is the centralizer of C in G, it is clear that, with this identification of $X^*(S)$ and $X^*(\mathsf{S})$, its k-root system with respect to S contains the $\mathfrak{f}$-root system of M_x with respect to S. Hence, the semi-simple k-rank of M is greater than or equal to the semi-simple $\mathfrak{f}$-rank of M_x. The latter equals $\mathfrak{f}$-rank M_x-dim $\mathsf{C}=k$-rank $G-$dim C. On the other hand, as the k-split torus C is contained in the center of M, the semi-simple k-rank of M is at most equal to k-rank $G-$dim C. We conclude from these observations that the semi-simple k-rank of M coincides with the semi-simple $\mathfrak{f}$-rank of M_x. As the maximal reductive quotient of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated with the parahoric subgroup $\mathcal{M}_x$ of $\mathcal{M}$ clearly contains M_x, both the assertions of the proposition are now obvious. (Note that the semi-simple $\mathfrak{f}$-rank of the maximal reductive quotient of the reduction mod $\mathfrak{p}$ of the $\mathfrak{o}$-group scheme associated to a parahoric subgroup of $\mathcal{M}$ is at most the semi-simple k-rank of M and the parahoric subgroup is maximal if and only if the equality holds; see, for example, [T:3.5].) $\square$

6.5. Fix a minimal K-type $(\mathcal{G}_x, \sigma)$ of depth zero. Part (2) of the preceding proposition allows us to naturally view σ as a cuspidal representation of $\mathcal{M}_x/\mathcal{M}_x^+$. Let $\mathcal{F}_x$ be the normalizer of $\mathcal{M}_x$ in $\mathcal{M}$. Then $\mathcal{F}_x$ is compact modulo the center of $\mathcal{M}$. Let $\mathcal{E}(\sigma)$ be the collection of irreducible representations (up to equivalence) of $\mathcal{F}_x$ which contain σ on restriction to $\mathcal{M}_x$. Note that each τ in $\mathcal{E}(\sigma)$ is finite dimensional since $\mathcal{F}_x$ is compact modulo its center. $\mathcal{E}(\sigma)$ consists precisely of the irreducible representations (of $\mathcal{F}_x$) contained in $\mathrm{Ind}_{\mathcal{M}_x}^{\mathcal{F}_x}\sigma$. One can partition $\mathcal{E}(\sigma)$ into finitely many equivalence classes by placing τ_1 and τ_2 in the same equivalence class if there exists an unramified quasicharacter χ of $\mathcal{M}$ such that $\tau_2 = \tau_1 \otimes \chi$.

6.6. Proposition. *Given $\tau \in \mathcal{E}(\sigma)$, the representation $\pi = \mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau$ is an irreducible absolutely cuspidal representation of $\mathcal{M}$. Moreover any irreducible representation of $\mathcal{M}$ which contains σ on restriction to $\mathcal{M}_x$ is isomorphic to c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau$ for some $\tau \in \mathcal{E}(\sigma)$. If χ is a one-dimensional quasicharacter of $\mathcal{M}$, the c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}}(\tau\otimes\chi) = ($c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau)\otimes\chi$.*

Proof. The assertion is a folklore theorem. A published proof appears in [Mr1]. The following proof is perhaps simpler. Recall [Sh:§3] that the compactly induced representation $\pi = \text{c-Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau$ is an irreducible (and therefore absolutely cuspidal) representation if and only if the Hecke algebra $\mathcal{H} := \mathcal{H}(\mathcal{M}//\mathcal{F}_x, \tau)$ of $\mathrm{End}(V_\tau)$-valued τ-spherical functions on $\mathcal{M}$ is one-dimensional. Given a double coset $\mathcal{F}_x m \mathcal{F}_x$ ($m \in \mathcal{M}$), let $\mathcal{H}(\mathcal{F}_x m\mathcal{F}_x, \tau)$ be the subspace of functions $f \in \mathcal{H}$ such that the support $\mathrm{supp}(f)$ of f is contained in the double coset $\mathcal{F}_x m\mathcal{F}_x$. The vector space $\mathcal{H}$ is a direct sum of the subspaces $\mathcal{H}(\mathcal{F}_x m\mathcal{F}_x, \tau)$. Because σ is a cuspidal representation, a necessary condition for the subspace $\mathcal{H}(\mathcal{F}_x m\mathcal{F}_x, \tau)$ to be nonzero is that the natural maps from $\mathcal{M}_x \cap m\mathcal{M}_x m^{-1} = \mathcal{M}_x \cap \mathcal{M}_{mx}$ to $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{mx}(\mathfrak{f})$ be surjective. (Note that for any m in $\mathcal{M}$, the image of $\mathcal{M}_x \cap \mathcal{M}_{mx}$ in $\mathsf{M}_x(\mathfrak{f})$ (resp. $\mathsf{M}_{mx}(\mathfrak{f})$) is the group of $\mathfrak{f}$-rational points of a parabolic $\mathfrak{f}$-subgroup of M_x (resp. M_{mx}) and the image of $\mathcal{M}_x \cap \mathcal{M}_{mx}^+$ (resp. $\mathcal{M}_x^+ \cap \mathcal{M}_{mx}$) is the group of $\mathfrak{f}$-rational points of the unipotent radical of this parabolic $\mathfrak{f}$-subgroup. This can be seen using the affine root system of M relative to a maximal K-split torus such that the corresponding apartment in the Bruhat–Tits building of M/K contains both x and mx.) Since the point x corresponds to a maximal parahoric subgroup of $\mathcal{M}$, the latter condition on surjectivity implies that the element m must fix the point x, i.e. $m \in \mathcal{F}_x$. The only double coset $\mathcal{F}_x m\mathcal{F}_x$ for which $\mathcal{H}(\mathcal{F}_x m\mathcal{F}_x, \tau)$ is nonzero is therefore the trivial double coset $\mathcal{F}_x$. Thus $\mathcal{H}$ is one-dimensional.

To prove the second assertion, suppose (π, V_π) is an irreducible representation of $\mathcal{M}$ which contains σ. An application of [BZ:2.28], shows there exists a $\tau \in \mathcal{E}(\sigma)$ such that $\mathrm{Hom}_{\mathcal{M}}(\text{c-Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau, V_\pi)$ is nonzero. Since $\text{c-Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau$ and V_π are both irreducible, they must be isomorphic.

Finally, it is obvious that $\text{c-Ind}_{\mathcal{F}_x}^{\mathcal{M}}(\tau\otimes\chi) = (\text{c-Ind}_{\mathcal{F}_x}^{\mathcal{M}}\tau)\otimes\chi$ if χ is a one-dimensional quasicharacter. $\square$

Now to be able to determine all the irreducible absolutely cuspidal representations of depth zero (see Proposition 6.8 below), we show that Lemma 4.7 of [B1], which is formulated only for the trivial representation of an Iwahori subgroup, can be formulated and proved for an arbitrary minimal K-type $(\mathcal{G}_x, \sigma)$. In [B1], the proof of Lemma 4.7 is based on the invertibility of certain elements in the Iwahori Hecke algebra. Our proof of the analogue for an arbitrary minimal K-type $(\mathcal{G}_x, \sigma)$ relies instead on Proposition 6.1.

Let $\mathcal{G}_x$ be a *nonmaximal* parahoric subgroup of $\mathcal{G}$. We fix a maximal k-split torus S so that the apartment $A(S)$ contains the point x. Let C be the subtorus of S as in 6.3. Let M be the centralizer of C in G and $\mathcal{M} = M(k)$. As $\mathcal{G}_x$ is not maximal, M is a proper Levi subgroup of G. Let $\mathcal{P} = \mathcal{M}\mathcal{N}$ and $\mathcal{P}' = \mathcal{M}\mathcal{N}'$ be two opposite parabolic subgroups of $\mathcal{G}$ with $\mathcal{M}$ as their common Levi factor. With these notations we have:

6.7. Proposition. *Let (π, V_π) be an irreducible admissible representation of $\mathcal{G}$. Then*

$$J_{\mathcal{N}} : V_\pi^{\mathcal{G}_x^+} \longrightarrow J_{\mathcal{N}}(V_\pi)^{\mathcal{M}_x^+}$$

is an isomorphism.

Proof. If α is a character of C and $\gamma : GL_1 \to C$ is a 1-parameter subgroup (of C), let $\langle \alpha, \gamma \rangle$ be the integer which satisfies $\alpha(\gamma(t)) = t^{\langle \alpha, \gamma \rangle}$. We fix a 1-parameter subgroup γ of C so that $\langle \alpha, \gamma \rangle > 0$ for every root α of C in the Lie algebra $\mathfrak{n}$ of $\mathcal{N}$, and consider the ray $x(t) = x + t\gamma$ $(t \geqq 0)$, contained in the apartment $A(S)$, emanating from the point x in the direction γ. It is easy to see that

(1) $\mathcal{M} \cap \mathcal{G}_{x(t)} = \mathcal{M}_x$ for all t.

(2) Set $\mathcal{N}_{x(t)} = \mathcal{N} \cap \mathcal{G}_{x(t)}$ and $\mathcal{N}'_{x(t)} = \mathcal{N}' \cap \mathcal{G}_{x(t)}$. If $t' \geqq t$, then

$$\mathcal{N}_{x(t')} \supset \mathcal{N}_{x(t)} \quad \text{and} \quad \mathcal{N}'_{x(t')} \subset \mathcal{N}'_{x(t)}.$$

(3) Any compact subgroup of $\mathcal{N}$ is contained in $\mathcal{N}_{x(t)}$ for t sufficiently large.

(4) There is a sequence $0 = t_0 < t_1 < t_2 \cdots < t_i < \ldots$ tending to ∞ so that $\mathcal{G}_{x(t)}$ is constant on the open intervals $t_{i-1} < t < t_i$ $(i > 1)$ as well as the interval $0 \leqq t < t_1$. On these intervals, $\mathcal{G}_{x(t)}$ is a parahoric subgroup which is associate to $\mathcal{G}_x$.

We shall denote the parahoric subgroup $\mathcal{G}_{x(t_i)}$ by $\mathcal{G}_i$ and its pro-nil radical $\mathcal{G}^+_{x(t_i)}$ by $\mathcal{G}_i^+$. The connected reductive $\mathfrak{f}$-group associated with $\mathcal{G}_i$ will be denoted by M_i. For $i \geqq 1$, $\mathcal{G}_{i^-}$ will denote the parahoric subgroup $\mathcal{G}_{x(t)}$, $t_{i-1} < t < t_i$ and let $\mathcal{N}_{i^-} = \mathcal{N} \cap \mathcal{G}_{i^-}$, $\mathcal{N}'_{i^-} = \mathcal{N}' \cap \mathcal{G}_{i^-}$. Note that for all $i \geqq 1$,

$$\mathcal{G}_{i^-} = \mathcal{N}'_{i^-}\mathcal{M}_x\mathcal{N}_{i^-},$$

and

$$\mathcal{G}_i^+ = \mathcal{N}'_{i+1^-}\mathcal{M}_x^+\mathcal{N}_{i^-}.$$

(5) The parahoric subgroup $\mathcal{G}_i$ contains both $\mathcal{G}_{i^-}$ and $\mathcal{G}_{i+1^-}$ as proper subgroups. The image of $\mathcal{G}_{i^-}$ in $\mathsf{M}_i(\mathfrak{f})$ is the group of $\mathfrak{f}$-rational points of a parabolic $\mathfrak{f}$-subgroup

P_i' of M_i and the image of $\mathcal{G}_{i+1^-}$ is the group of $\mathfrak{f}$-rational points of a parabolic $\mathfrak{f}$-subgroup P_i; P_i and P_i' are opposed and the Levi subgroup $\mathsf{P}_i \cap \mathsf{P}_i'$ is $\mathfrak{f}$-isomorphic to M_x. We note for later use that $\mathcal{N}_{i^-}\mathcal{M}_x^+\mathcal{N}'_{i^-}$ is a subgroup of $\mathcal{G}_i$ containing $\mathcal{G}_i^+$, in fact it is the inverse image in $\mathcal{G}_i$ of the unipotent radical of $\mathsf{P}_i'(\mathfrak{f})$.

Let $v \in V_\pi^{\mathcal{G}_x^+}$ be a nonzero vector. We claim $J_\mathcal{N}(v) \neq 0$. To prove this claim, we define vectors $v_i, i \geqq 1$, inductively as follows: Let $v_1 = v$ and for $i > 1$, let

$$v_i \;=\; \int_{\mathcal{N}_{i^-}} \pi(n) v_{i-1}\, dn \quad .$$

We shall show by induction that for all i, (i) $v_i \neq 0$, (ii) v_i is fixed under $\mathcal{G}_i^+$ $(\supset \mathcal{M}_x^+)$ and $\mathcal{N}'_{i^-}$, and (iii) v_i is a nonzero multiple of $\int_{\mathcal{N}_{i^-}} \pi(n) v\; dn$. From its definition it is clear that v_i is fixed under $\mathcal{N}_{i^-}$. Moreover since v_{i-1} is fixed under $\mathcal{M}_x^+\mathcal{N}'_{i-1^-} (\supset \mathcal{M}_x^+\mathcal{N}'_{i^-})$ and $\mathcal{N}_{i^-}\mathcal{M}_x^+\mathcal{N}'_{i^-}$ is a compact subgroup, v_i is fixed under the latter. Since v_{i-1} is fixed under $\mathcal{G}_{i-1}^+$ and also under $\mathcal{N}'_{i-1^-}$, and the latter projects onto the unipotent radical of $\mathsf{P}'_{i-1}(\mathfrak{f})$, whereas $\mathcal{N}_{i^-}$ projects onto the unipotent radical of $\mathsf{P}_{i-1}(\mathfrak{f})$, Proposition 6.1 (2) implies that if v_{i-1} is nonzero then so is v_i. Assertion (iii) can be proved easily using Fubini's theorem and the bi-invariance of the Haar measure on the unipotent group $\mathcal{N}$. It implies that for every $i \geqq 1$,

$$\int_{\mathcal{N}_{i^-}} \pi(n) v\; dn \neq 0.$$

As any compact subgroup of $\mathcal{N}$ is contained in $\mathcal{N}_{i^-}$ for i large, we conclude now that $J_\mathcal{N}(v) \neq 0$. Since v was assumed to be an arbitrary element of $V_\pi^{\mathcal{G}_x^+}$, it follows that $J_\mathcal{N}$ is an injection of $V_\pi^{\mathcal{G}_x^+}$ into $J_\mathcal{N}(V_\pi)^{\mathcal{M}_x^+}$. By Theorem 2.2, $J_\mathcal{N}$ is also a surjection; whence it is an isomorphism. $\square$

6.8. Proposition. *Every irreducible depth zero absolutely cuspidal representation of* $\mathcal{G}$ *has the form* c-$Ind_{\mathcal{F}_x}^{\mathcal{G}}\tau$ *for some maximal parahoric subgroup* $\mathcal{G}_x$ *of* $\mathcal{G}$*, cuspidal representation* σ *of* $\mathsf{M}_x(\mathfrak{f})$ *inflated to* $\mathcal{G}_x$*, and* $\tau \in \mathcal{E}(\sigma)$*.*

Let (π, V_π) be an irreducible absolutely cuspidal representation of $\mathcal{G}$ which contains the minimal K-type $(\mathcal{G}_x, \sigma)$. Let $\mathcal{M}$ be the Levi subgroup of $\mathcal{G}$ associated to the parahoric $\mathcal{G}_x$. We assert that $\mathcal{G}_x$ is a *maximal* parahoric subgroup of $\mathcal{G}$ (or, equivalently, $\mathcal{M} = \mathcal{G}$). If $\mathcal{M} \neq \mathcal{G}$, let $\mathcal{N}$ be the unipotent radical of a parabolic subgroup of $\mathcal{G}$ with Levi subgroup $\mathcal{M}$; according to the preceding proposition, $J_\mathcal{N}(V_\pi) \neq 0$; this contradicts the fact that π is absolutely cuspidal and we conclude that $\mathcal{G}_x$ is maximal and $\mathcal{M} = \mathcal{G}$. Now the proposition follows from Proposition 6.6.

Morris has indicated in the introduction of [Mr2] a different proof of Proposition 6.8 based on Hecke algebras.

6.9. We are now prepared to show for an arbitrary minimal K-type $(\mathcal{G}_x, \sigma)$ of depth zero, the analogues of the well known results of Borel [B1] that

(1) Any subquotient of an unramified principal series contains a nonzero Iwahori fixed vector.

(2) Any irreducible admissible representation which has a nonzero Iwahori fixed vector occurs as a subquotient of an unramified principal series.

6.10. Fix a minimal K-type $(\mathcal{G}_x, \sigma)$ of depth zero. Let M be the Levi subgroup attached to $\mathcal{G}_x$ in 6.3. Let $\mathcal{M} = M(k)$, and let $\mathcal{P} = \mathcal{M}\mathcal{N}$ be a parabolic subgroup of $\mathcal{G}$ containing $\mathcal{M}$ as a Levi subgroup. For $\tau \in \mathcal{E}(\sigma)$, let $\theta =$c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}} \tau$. Form the generalized principal series $\mathrm{I}(\mathcal{P}, \theta) = \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}} \theta$.

6.11. Theorem.

(1) *Any subquotient π of $I(\mathcal{P}, \theta)$ is generated by its $(\mathcal{G}_x, \sigma)$-isotypic subspace.*
(2) *Conversely, any irreducible admissible representation (π, V_π) whose $(\mathcal{G}_x, \sigma)$-isotypic subspace is nonzero appears as a subquotient of $I(\mathcal{P}, \theta)$ where $\theta =$ c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}} \tau$ for some $\tau \in \mathcal{E}(\sigma)$.*

Proof. To prove (1), it is sufficient to prove that any irreducible subquotient π of $\mathrm{I}(\mathcal{P}, \theta)$ is generated by its $(\mathcal{G}_x, \sigma)$-isotypic subspace. Therefore, we assume π is an irreducible subquotient. We apply Theorem 2.5 to find a parabolic $\mathcal{Q} = \mathcal{M}\mathcal{U}$ which is $\mathcal{M}$-associate of $\mathcal{P} = \mathcal{M}\mathcal{N}$ and for which π is an irreducible subrepresentation of $\mathrm{I}(\mathcal{Q}, \theta)$. The Jacquet module $J_{\mathcal{U}}(V_\pi)$ of π has V_θ as a quotient (2.3). The subgroup $\mathcal{G}_x^+$ has the Iwahori factorization with respect to the parabolic subgroup $\mathcal{Q}$. So the map

$$V_\pi^{\mathcal{G}_x^+} \longrightarrow V_\theta^{\mathcal{M}_x^+}$$

is a surjection (2.2). According to Proposition 6.4, $\mathcal{G}_x/\mathcal{G}_x^+ \cong \mathcal{M}_x/\mathcal{M}_x^+$. So it follows that $(\mathcal{G}_x, \sigma)$ occurs in π and thus must generate it.

To prove part (2), we use the following lemma.

6.12. Lemma. *Let $\mathcal{G}_x$ be a nonmaximal parahoric subgroup of $\mathcal{G}$. Fix an apartment $A(S)$ and a Levi subgroup $\mathcal{M}$ as in section 6.3 so that $x \in A(S)$ and $\mathcal{M}_x/\mathcal{M}_x^+ \cong \mathcal{G}_x/\mathcal{G}_x^+$. Let $y \in A(S)$ be such that $\mathcal{M}_y/\mathcal{M}_y^+ \cong \mathcal{G}_y/\mathcal{G}_y^+$ and there is a minimal K-type $(\mathcal{G}_y, \xi)$ associate to the minimal K-type $(\mathcal{G}_x, \sigma)$, then there exists a $g \in \mathcal{G}$ so that*

(1) *$gy \in A(S)$*
(2) *g normalizes $\mathcal{M}$*
(3) *the natural maps from $\mathcal{G}_x \cap \mathcal{G}_{gy}$ to $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{gy}(\mathfrak{f})$ are surjective and ξ^g is equivalent to σ*
(4) *the natural maps from $\mathcal{M}_x \cap \mathcal{M}_{gy}$ to $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{gy}(\mathfrak{f})$ are surjective.*

Proof. We begin by recalling some notation from 6.3. C is the maximal $\mathfrak{f}$-split torus contained in the center of M_x and C is its lift to S. The Levi M is the centralizer of C in G, and $\mathcal{M}$ is the group of k-rational points of M. We identify the character group $X^*(S)$ of S with the character group $X^*(\mathsf{S})$ of S in terms of their natural identification. The condition $\mathcal{M}_y/\mathcal{M}_y^+ \cong \mathcal{G}_y/\mathcal{G}_y^+$ implies that the root system $\Phi(\mathsf{S}, \mathsf{M}_y)$ of M_y is contained in the root system $\Phi(S, M)$ of M. Let $g \in \mathcal{G}$ be an element so that $\mathcal{G}_x \cap \mathcal{G}_{gy}$ projects onto both $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{gy}(\mathfrak{f})$ and ξ^g is equivalent to σ. As $\mathcal{G}_x$ acts transitively on the set of apartments (in the building of $\mathcal{G}$) containing x, after replacing g with a suitable element in

$\mathcal{G}_x g$, we assume that gy lies in the apartment $A(S)$. Since y also lies in the same apartment, we may (and do) assume, after replacing g with an element in $g\mathcal{G}_y$, that g normalizes the maximal k-split torus S.

It is clear that g carries the root system $\Phi(\mathsf{S}, \mathsf{M}_y)$ onto the root system $\Phi(\mathsf{S}, \mathsf{M}_{gy})$ of M_{gy} (both these root systems are considered as subsets of the root system $\Phi(S, G)$ of G in terms of the identification of $X^*(\mathsf{S})$ with $X^*(S)$). Now since $\mathcal{G}_x \cap \mathcal{G}_{gy}$ projects onto both $\mathsf{M}_x(\mathfrak{f})$ and $\mathsf{M}_{gy}(\mathfrak{f})$, it follows that the root system $\Phi(\mathsf{S}, \mathsf{M}_x)$ of M_x equals the root system $\Phi(\mathsf{S}, \mathsf{M}_{gy})$. Thus g carries $\Phi(\mathsf{S}, \mathsf{M}_y)$ onto $\Phi(\mathsf{S}, \mathsf{M}_x)$. Now as the root system $\Phi(\mathsf{S}, \mathsf{M}_y)$ is contained in the root system $\Phi(S, M)$ and C is the identity component of the intersection of kernels of the roots in $\Phi(S, M)$, whereas C, and so also C, is the identity component of the intersection of kernels of roots in $\Phi(\mathsf{S}, \mathsf{M}_x)$, we conclude that C is the identity component of the intersection of the kernels of roots in $\Phi(\mathsf{S}, \mathsf{M}_y)$ and so g normalizes C. Hence, it also normalizes the centralizer M of C. In view of the fact that $\Phi(\mathsf{S}, \mathsf{M}_{gy}) = \Phi(\mathsf{S}, \mathsf{M}_x) \subset \Phi(S, M)$, assertion (4) follows from the surjectivity statement in (3). □

We turn to the converse (2). Suppose π contains the minimal K-type $(\mathcal{G}_x, \sigma)$. If $\mathcal{G}_x$ is a maximal parahoric subgroup of $\mathcal{G}$, then $\mathcal{M} = \mathcal{G}$ and according to Proposition 6.6, π is an absolutely cuspidal representation and it is isomorphic to c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{G}} \tau$ for some $\tau \in \mathcal{E}(\sigma)$.

Suppose now that $\mathcal{G}_x$ is not a maximal parahoric subgroup. Let $\mathcal{M}$ and $\mathcal{P}$ be as in 6.10. Let $(\theta', V_{\theta'})$ be an irreducible quotient of $J_{\mathcal{N}}(V_\pi)$. Note that according to Theorem 5.2 the depth of θ' is zero. Let $(\mathcal{M}_y, \chi)$ be a minimal K-type for θ', with $y \in A(S)$. In view of Lemma 5.1, we may (and we shall) assume that $\mathcal{M}_y/\mathcal{M}_y^+ \cong \mathcal{G}_y/\mathcal{G}_y^+$. It follows at once from 2.2 and the fact that $\mathcal{G}_y/\mathcal{G}_y^+ \cong \mathcal{M}_y/\mathcal{M}_y^+$ that there exists a minimal K-type $(\mathcal{G}_y, \xi)$ for π such that the restriction of ξ to $\mathcal{M}_y$ is χ. Now according to Theorem 3.5 the minimal K-types $(\mathcal{G}_x, \sigma)$ and $(\mathcal{G}_y, \xi)$ are associates of each other and hence there exists an element $g \in \mathcal{G}$ satisfying the four conditions listed in Lemma 6.12.

Let $\mathcal{P}' = g\mathcal{P}g^{-1}$ and $\mathcal{N}' = g\mathcal{N}g^{-1}$ so that $\mathcal{P}' = \mathcal{M}\mathcal{N}'$. Consider the Jacquet functor $J_{\mathcal{N}'}$. The representation $\theta = \theta'^g$ of $\mathcal{M}$ given by $\theta(m) = \theta'(g^{-1}mg)$ is an irreducible quotient of $J_{\mathcal{N}'}(V_\pi)$. Furthermore, θ contains the minimal K-type $(\mathcal{M}_{gy}, \chi^g)$ and therefore must also contain $(\mathcal{M}_x, \sigma)$ by 6.2 applied to $\mathcal{M}$. It now follows from Proposition 6.6 that θ is equivalent to c-$\mathrm{Ind}_{\mathcal{F}_x}^{\mathcal{M}} \tau$ for some $\tau \in \mathcal{E}(\sigma)$. Frobenius reciprocity (2.3) tells us that π is a subrepresentation of $\mathrm{I}(\mathcal{P}', \theta)$. By Theorem 2.5, $\mathrm{I}(\mathcal{P}', \theta)$ and $\mathrm{I}(\mathcal{P}, \theta)$ have the same composition factors. In particular, π is a subquotient of $\mathrm{I}(\mathcal{P}, \theta)$. □

6.13. Let $(\mathcal{G}_x, \sigma)$ be a minimal K-type of depth zero. Let $\mathcal{C} = \mathcal{C}(\mathcal{G}_x, \sigma)$ be the full subcategory of the category of admissible representations of $\mathcal{G}$ of finite length whose objects are those admissible representations π of $\mathcal{G}$ all of whose subquotients are generated by their $(\mathcal{G}_x, \sigma)$-isotypic subspaces. Denote by Θ_σ the character of σ. Let e_σ be the idempotent element in the algebra $\mathcal{H}(\mathcal{G})$ of compactly supported functions on $\mathcal{G}$ which is supported on $\mathcal{G}_x$ and given there by $e_\sigma(g) = \frac{\deg\sigma}{\mathrm{vol}(\mathcal{G}_x)} \overline{\Theta}_\sigma(g)$. In particular, if (π, V_π) is a representation of $\mathcal{G}_x$, then $\pi(e_\sigma)$ is projection onto the σ-isotypic subspace V_π^σ of V_π. In fact, $V_\pi^\sigma = \pi(e_\sigma)(V_\pi)$ is naturally a module for the algebra $\mathcal{H}(\mathcal{G}, e_\sigma) = e_\sigma \mathcal{H}(\mathcal{G}) e_\sigma$ (an algebra in which e_σ is the identity) and the process $\mathcal{F}$ of taking σ-isotypic subspace is a functor

$$V_\pi \xrightarrow{\mathcal{F}} V_\pi^\sigma$$

from $\mathcal{C}(\mathcal{G}_x, \sigma)$ to the category $\mathcal{C}(\mathcal{H}(\mathcal{G}, e_\sigma))$ of finite dimensional representations of $\mathcal{H}(\mathcal{G}, e_\sigma)$.

The algebra $\mathcal{H}(\mathcal{G}, e_\sigma)$ is very simply related to the algebra $\mathcal{H}(\mathcal{G}//\mathcal{G}_x, \check{\sigma})$, where $\check{\sigma}$ denotes the contragredient of σ. Let $M(\sigma) = e_\sigma \mathcal{H}(\mathcal{G}_x) e_\sigma$, an ideal of $\mathcal{H}(\mathcal{G}_x)$. Note that $M(\sigma)$ as an algebra (with identity e_σ) is simple of dimension $\deg(\sigma)^2$, and that if β is an irreducible representation of $\mathcal{G}_x$, then $\beta(M(\sigma)) = 0$ unless β is equivalent to σ. As observed in [BK1], $\mathcal{H}(\mathcal{G}//\mathcal{G}_x, \check{\sigma}) \otimes_{\mathbb{C}} M(\sigma)$ is canonically isomorphic to $\mathcal{H}(\mathcal{G}, e_\sigma)$.

6.14. Corollary.

(1) *If (π, V_π) is an admissible representation which is generated by its σ-isotypic subspace V_π^σ, then (π, V_π) belongs to $\mathcal{C}(\mathcal{G}_x, \sigma)$*

(2) *The functor*

$$\mathcal{F} : \mathcal{C}(\mathcal{G}_x, \sigma) \longrightarrow \mathcal{C}(\mathcal{H}(\mathcal{G}, e_\sigma))$$

is an equivalence of categories.

Proof. Both (1) and (2) follow from Theorem 4.4 in [BK2] as immediate consequences of Theorem 6.11. □

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