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LOCAL CHARACTER EXPANSIONS

Dan Barbasch and Allen Moy

Abstract. In this paper the authors develop a method to compute the local character expansion of a depth zero representation of a p–adic group. The main idea is to use the generalized Gelfand–Graev characters for finite groups as test functions to plug into the character formula. This is possible due to results of Waldspurger on the validity of the local character expansion in a large enough neighborhood of the identity. The method leads to a classification of the unipotent orbits in terms of parahoric subalgebras.

Dans cet article, les auteurs développent une méthode pour calculer l'expansion locale d'un caractère d'une representation de profondeur zero. L'idée principale est d' utiliser les caractères de Gelfand–Graev généralisés des groupes finis. C'est faisable grace aux resultats the J.–L. Waldspurger qui montre que le developpement du caractère est valide sur un voisinage suffisament grand. La méthode donne une classification des orbites unipotentes en termes des sous-algebres parahoriques.

1. Introduction

Suppose k is a nonarchimedean local field of characteristic zero and G is a reductive group defined over k with Lie algebra $\mathrm{Lie}(G)$. Let $G(k)$ be the k–rational points of G and Θ_π the character of an irreducible admissible representation π of $G(k)$. Harish–Chandra proved that Θ_π can be represented as a locally constant integrable function on the regular set. Generalizing a result of Howe for $\mathrm{GL}(n)$ [Ho], Harish–Chandra also showed that Θ_π is a linear combination of Fourier transforms of nilpotent orbits in a sufficiently small neighborhood of zero in $\mathfrak{g} = \mathrm{Lie}(G)(k)$, *i.e.* there exist constants $c_{\mathcal{O}}(\pi)$, one for each nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ such that

$$\Theta_\pi(\ \exp X\ ) = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X)$$

for $X \in \mathfrak{g}$, a regular element sufficiently close to 0. Much qualitative and quantitative information on the representation π can be gleaned from the Harish-Chandra–Howe local character expansion. For instance, the asymptotic growth of the space of vectors of π fixed by parahoric filtration subgroups $\mathcal{P}_{x,r_k}$ as $k \to \infty$ is controlled by the orbits $\mathcal{O}$ of maximal dimension with $c_{\mathcal{O}} \neq 0$. These same orbits also relate to the existence of various generalized Whittaker models for π ([MW]). Another example of the quantitative information in the local character expansion occurs when π is a discrete series representation. Here, the formal degree of π equals $(-1)^r c_0(\pi)$ where r is the split rank of G over k and $c_0(\pi)$ is the coefficient of the zero orbit [Ro].

Two natural questions to ask in regard to the Harish-Chandra–Howe local character expansion are

(1) what can be said about the size of the neighborhood in $\mathfrak{g}$ on which the local character expansion is valid and
(2) what techniques are there to compute the coefficients $c_{\mathcal{O}}$?

In regard to the first question, Waldspurger [Wa] has proved a conjectured sharpening (due to Hales [Ha]) of the results of Howe and Harish–Chandra. A representation π is defined to have depth zero, if there exists a parahoric $\mathcal{P}$ so that π possesses non zero vectors fixed by $\mathcal{P}_+$, the maximal normal pro-p-subgroup of $\mathcal{P}$. Under mild conditions, Waldspurger has shown that if G is essentially a group of classical type and π is a depth zero representation, then the local character expansion is valid on each neighborhood $\log(\mathcal{Q}_+)$ ($\mathcal{Q}$ a parahoric subgroup). In particular, Waldspurger's theorem gives a very complete and satisfactory answer to the first question for representations of depth zero.

In regards to the second question, one type of answer can be found in [MW] in terms of generalized Whittaker models.

It is the purpose of this paper to show how Waldspurger's results in [Wa] also leads to a solution of the second question along very different lines from [MW]. We show that the validity of the local character expansion for depth zero representations, can be used as a trace formula from which one can completely determine the coefficients $c_{\mathcal{O}}(\pi)$ in terms of the depth zero K–types of π. Our method is to inflate the characters of generalized Gelfand–Graev representations for reductive groups over finite fields associated to parahoric subgroups of $G(k)$ and show that the inflations can be used as test functions in the local character expansion. In brief, if p is sufficiently large, then the Gelfand–Graev characters can be pulled back via the exponential map to the Lie algebra. The support of the Fourier tranforms of the Gelfand–Graev characters on the topologically nilpotent set in the Lie algebra is compatible with the partial order of nilpotent orbits in a finite field Lie algebra. In section 3 (Propositions 3.5 and 3.8) we give a classification of the nilpotent orbits of a Lie algebra $\mathfrak{g}$ in terms of associate classes of parahoric subalgebras $\mathfrak{p}_x$ and nilpotent orbits of $\mathfrak{p}_x/\mathfrak{p}_x^+$. This classification highlights properties of the cosets $N+\mathfrak{p}_x^+$ and their closure relations (Propositions 3.12 and 3.16) which had been noticed independently by Howe [Ho] for $\mathrm{GL}(n)$ at an earlier time. These results generalize the results in [Wb] (rather [Wc] which only seems to be available in preprint form). While [Wc] proceeds case by case and deals only with the classical cases, our methods are quite general and treat all cases at the same time.

The results in section 3 lead to a recursive method for calculating the coefficients $c_{\mathcal{O}}(\pi)$ via the partial ordering on the orbits $\mathcal{O}$'s.

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2. Generalized Gelfand–Graev representations

2.1. We summarize the construction of the generalized Gelfand–Graev representations. Let $\mathbb{F}_q$ be a finite field of characteristic p. Fix an algebraic closure $\overline{\mathbb{F}_q}$ of $\mathbb{F}_q$. Let G be a simple adjoint group defined over $\mathbb{F}_q$ and let $\mathfrak{g}$ be its Lie algebra. We identify G (*resp.* $\mathfrak{g}$) with its $\overline{\mathbb{F}_q}$ points. Denote the Frobenius actions on G and $\mathfrak{g}$ by F. In particular, the F-fixed points G^F and $\mathfrak{g}^F$ correspond to the $\mathbb{F}_q$–rational points of G and $\mathfrak{g}$ respectively. We recall some basic results from chapter 5 of [Ca] on the nilpotent elements in $\mathfrak{g}$. Let $T \subset G$ be a maximal torus in G and let $\Delta(G,T)$ (*resp.* $\Phi(G,T) = \{\alpha_1, \dots, \alpha_r\}$) be the roots (*resp.* simple roots). The height $ht(\alpha)$ of a root $\alpha = \sum_i n_i\alpha_i$ is $ht(\alpha) = \sum_i n_i$. Set $m = \max_\alpha ht(\alpha)$. The relation $m = h-1$, where h is the Coxeter number, holds. Consider the inclusion $\mathfrak{g} \overset{ad}{\hookrightarrow} \mathfrak{gl}(\mathfrak{g})$.

(1) If $N \in \mathfrak{g}$ is nilpotent, then $ad(N)^{2h-1} = 0$, [Ca:5.5.2]. In particular, if $p > 2h-1$, then the exponential map from $\mathfrak{gl}(\mathfrak{g})$ into $\mathrm{GL}(\mathfrak{g})$ is defined on the nilpotent elements of $\mathfrak{g}$.

(2) Suppose $N \in \mathfrak{g}$ is nilpotent, non zero and $ad(N)^n = 0$. If $p - 2 > n$, then the Jacobson–Morozov theorem is valid for N, [Ca:5.3.2], *i.e.* N determines an embedding

$$\phi_N : \mathfrak{sl}(2) \longrightarrow \mathfrak{g}$$

so that $\phi_N(e) = N$, where e, h, f are the usual elements in $\mathfrak{sl}(2)$. If N lies in $\mathfrak{g} = \mathfrak{g}^F$, then this $\mathfrak{sl}(2)$ can be taken to be defined over $\mathbb{F}_q$. Furthermore, two embeddings ϕ_N and $\phi_{N'}$ are equivalent under the adjoint action of G^F if and only if N and N' belong to the same G^F conjugacy class.

(3) In case (2), if $p > 3(h-1)$, then $\mathfrak{g}$ is a completely reducible $\mathfrak{sl}(2)$ module under the adjoint action. These modules are "the same" as over $\mathbb{C}$ ([Ca:5.5.5]).

Hence, following [Ca], we assume that

$$p > 3(h-1). \tag{2.1.1}$$

Then both the log and exp maps are defined on the unipotent and nilpotent elements of G and $\mathfrak{g}$ respectively and properties (2) and (3) are valid. Furthermore, by the theorem of Dynkin–Kostant, there are finitely many nilpotent conjugacy classes in $\mathfrak{g}$.

2.2. Let N be a nilpotent element in $\mathfrak{g}^F$ (with associated $\mathfrak{sl}(2)$–triple $N = e, h, f$). Let

$$\begin{aligned} \mathfrak{g}(i) &= \{\ X \in \mathfrak{g} \mid [h, X] = iX\ \} \qquad i \in \mathbb{Z} \\ \mathfrak{m} &= \mathfrak{g}(0) \\ \mathfrak{u}_+ &= \underset{i>0}{\oplus}\, \mathfrak{g}(i) \\ \mathfrak{u}_- &= \underset{i<0}{\oplus}\, \mathfrak{g}(i) \\ \mathfrak{p} &= \mathfrak{m} + \mathfrak{u}_+ \end{aligned}$$

and let M (*resp.* U, P, *etc.*) be the algebraic group corresponding to $\mathfrak{m}$ (*resp.* $\mathfrak{u}_+$, $\mathfrak{p}$, *etc.*). Via the Killing form, identify $\mathfrak{u}_-$ with the dual $\mathfrak{u}_+^*$. Let $\mathcal{O}_U = \mathcal{O}_U(-f)$ be the U^F coadjoint orbit of the element $-f$. Note that e and $-f$ are conjugate via $\mathrm{SL}(2)^F$ but that e and f need not be conjugate.

2.3. By Kirillov theory for nilpotent groups, the coadjoint orbit $\mathcal{O}_U$ determines a $|\mathcal{O}_U|^{\frac{1}{2}}$–dimensional representation η_N of U^F. The induced representation

$$\Gamma_N = \mathrm{Ind}_{U^F}^{G^F}\ \eta_N$$

is called the generalized Gelfand–Graev representation of G^F associated to N. When N is regular, the Gelfand–Graev representation is the finite field analogue of a space of Whittaker vectors. Let γ_N denote the character of Γ_N. Various properties of γ_N are collected together in the following proposition.

2.4. Proposition. *(Kawanaka [Ka,Kb])*

(1) *γ_N depends only on the $Ad(G^F)$–orbit of N.*
(2) $|\mathcal{O}_U|^{\frac{1}{2}} = q^{\frac{(dim(U) - dim(Ad(P)N))}{2}}$
(3) $dim(\gamma_N) = |G^F| q^{-\frac{dim(Ad(G)N)}{2}}, \qquad dim(Ad(G)N) = dim(U) + dim(Ad(P)N)$
(4) *The support of γ_N is contained in the closure of $Ad(G)(\exp N)$.*

Let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}$. We see that the support of γ_N is contained in the set of unipotent elements and therefore via the exponential map, we can transfer γ_N to an $Ad(G^F)$–invariant function on the Lie algebra $\mathfrak{g}^F$. The function $\gamma_N \circ \exp$ is supported inside the closure of $\mathcal{O}(N)$. For ease of notation, when the context is clear, we use γ_N to denote the function on $\mathfrak{g}^F$ which is equal to $\gamma_N \circ \exp$ on the nilpotent elements and zero elsewhere. This will be done in the next section.

2.5. Fix ψ be a nonzero additive character of $\mathbb{F}_q$. Define the Fourier transform (with respect to the Cartan–Killing form $\langle\ ,\ \rangle$) of a function f on $\mathfrak{g}^F$ by

$$\mathrm{FT}_{\mathfrak{g}^F}(f)(x) = \hat{f}(x) = \sum_{x' \in \mathfrak{g}^F} \psi(\langle x, x' \rangle) f(x').$$

The Fourier transform of a function supported on $\mathcal{N}^F$, *e.g.* γ_N, will in general not have support contained in $\mathcal{N}^F$. The following two results of Lusztig give a general expression for the values of $\widehat{\gamma}_N$ at an arbitrary element $y \in \mathfrak{g}^F$ as well as more detailed information at a nilpotent element. Set

$$\Sigma = -f + C_{\mathfrak{g}}(e) \tag{2.5.1}$$

where $C_{\mathfrak{g}}(e)$ is the centralizer of $N = e$ in $\mathfrak{g}$ and let

$$\begin{aligned} r(N) &= \frac{1}{2}\dim\mathfrak{g}(1) + \dim\mathfrak{g}(i \geq 2) \\ &= \frac{1}{2}(\dim\mathfrak{g} - \dim(C_{\mathfrak{g}}(N))) \end{aligned} \tag{2.5.2}$$

2.6. Proposition. *(Lusztig [L] Prop.2.5.) For any $y \in \mathfrak{g}^F$*

$$\widehat{\gamma}_N(y) = q^{r(N)} \ \#\{ \ g \in G^F \ \mid Ad(g)(y) \in \Sigma \ \}$$

In particular, $\widehat{\gamma}_N$ is integer valued.

2.7. Proposition. *(Lusztig [L] Prop.6.13.) For $N' \in \mathcal{N}^F$, let $\mathcal{O}(N')$ be the $Ad(G)$–orbit of N' in $\mathfrak{g}$.*

1. *If $\widehat{\gamma}_N(N') \neq 0$, then N must lie in the closure of $\mathcal{O}(N')$,*
2. *if $N \in \mathcal{O}(N')$ and $\widehat{\gamma}_N(N') \neq 0$, then N' is in the G^F–orbit of N,*
3. *$\widehat{\gamma}_N(N) = q^{r(N)} \ \#C_G(N)^F$.*

Roughly speaking, Proposition 2.7 says that the Fourier transform inverts the support of a generalized Gelfand–Graev representation γ_N, *i.e.* $\gamma_N(N') = 0$ if N is not in the closure of $Ad(G)(N')$ in $\mathfrak{g}$; furthermore, if $\mathcal{O}(N) = \mathcal{O}(N')$, then $\gamma_N(N') \neq 0$ precisely if N' is G^F conjugate to N.

3. Transition to the p–adic group

3.1. Let k be a p-adic field, with ring of integers $\mathfrak{o}_k$ and prime ideal $\mathfrak{p}_k$. Let $\mathbb{F}_q = \mathfrak{o}_k/\mathfrak{p}_k$ be the residue field. We wish to lift generalized Gelfand–Graev characters to functions on the p-adic group. These functions will be supported on the topologically unipotent/nilpotent set. Let K be a fixed maximal unramfied extension of k. Let $\mathfrak{g} = \mathrm{Lie}(G)(K)$ (*resp.* $\mathfrak{g} = \mathrm{Lie}(G)(k)$) denote the K–rational (*resp.* k–rational) points of the Lie algebra of G. Let $\mathcal{B}_K$ and $\mathcal{B}_k$ be the Bruhat–Tits building of the group G over the fields K and k respectively. The Galois group $\mathrm{Gal}(K/k)$ acts on $\mathcal{B}_K$ so that the building $\mathcal{B}_k$ can be naturally identified as the fixed points of $\mathrm{Gal}(K/k)$. Given a point $x \in \mathcal{B}_K$, let P_x, $\mathfrak{g}_{x,0}$, $\mathfrak{g}_{x,0^+}$, $\mathfrak{g}_{x,r_i}$, *etc.* be the parahoric subgroup and various lattices associated to the point x as in [MPa]. The quotient $\mathfrak{m}_x = \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}$ is the Lie algebra of $\mathsf{M}_x = P_{x,0}/P_{x,0^+}$. If the point x lies in $\mathcal{B}_k$, then both $\mathfrak{m}_x$ and M_x are defined over $\mathbb{F}_q$. We assume now that $x \in \mathcal{B}_k$. Let $\mathcal{P}_x = P_x \cap G(k)$, and let $\mathfrak{g}_{x,r_i} = \mathfrak{g}_{x,r_i} \cap \mathfrak{g}$.

Let $\tau : G \longrightarrow \mathrm{GL}(V)$ be a representation of G such that the corresponding representation $T_\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(V(k))$ of the k–rational points $\mathfrak{g}$ of $\mathrm{Lie}(G)$ is faithful. The following lemma is elementary, *e.g.* see [Wa:III.3] for a proof.

3.2. Lemma. *If $T_\tau : \mathfrak{g} \longrightarrow \mathfrak{gl}(V(k))$ is faithful, and the characteristic p of k satisfies*

$$\frac{p-1}{dim(\tau)} > val_k(p), \tag{3.2.1}$$

then the exponential map from $\mathfrak{gl}(V(k))$ to $GL(V(k))$ is defined for any element $Y \in \mathfrak{g}_{x,0^+}$ ($x \in \mathcal{B}_k$).

We will mainly use this in the case when τ is the adjoint representation, in which case $\exp(\mathfrak{g}_{x,0^+}) = P_{x,0^+}$.

3.3. Let ψ be an additive character of the p–adic field which is trivial on $\wp$ but not on $\mathfrak{o}$. The character ψ induces naturally a nontrivial additive character of the residue field $\mathbb{F}_q = \mathfrak{o}/\wp$. We shall denote both characters by ψ. The Fourier transform

$$\mathrm{FT}_{\mathfrak{g}}(f)(x) = \widehat{f}(x) = \int_{\mathfrak{g}} \psi(\langle x, x' \rangle) f(x') dx'$$

is related to the finite field Fourier transform as follows. Given a function f on $\mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}$, we denote by $\widetilde{f}$ its lift to $\mathfrak{g}_{x,0}$. We have

$$\mathrm{FT}_{\mathfrak{g}}(\widetilde{f}) = \mathrm{vol}(\mathfrak{g}_{x,0^+}) \widetilde{\mathrm{FT}_{\mathfrak{m}_x^F}}(f). \tag{3.3.1}$$

We shall need to evaluate the nilpotent orbital integrals of the inflated functions $\widetilde{f}$. Suppose N' is a nilpotent element in $\mathfrak{g}$ and $\widetilde{f}$ is a function on $\mathfrak{g}$ which is supported on $\mathfrak{g}_{x,0}$ and constant on cosets of $\mathfrak{g}_{x,0^+}$. The orbital integral of $\widetilde{f}$ with respect to the orbit $\mathcal{O}' = \mathcal{O}(N')$ can be written as

$$\begin{aligned} \int_{\mathcal{O}'} \widetilde{f}(y) dy &= \sum_Y \int_{\mathcal{O}' \cap Y} f(Y) dy \\ &= {\sum_Y}' f(Y)\ \mu_{\mathcal{O}'}(\mathcal{O}' \cap Y) \end{aligned} \tag{3.3.2}$$

where $\sum$ is the sum over the cosets $Y = y + \mathfrak{g}_{x,0^+} \subset \mathfrak{g}_{x,0}$ and $\sum'$ is the sum over those cosets whose image in the Lie algebra $\mathfrak{m}_x = \mathfrak{m}_x^F = \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}$ is nilpotent. Here and in the later sections we let $\mu_{\mathcal{O}}$ denote a nonzero invariant measure supported on the orbit of $\mathcal{O}$. The normalization will not be so important for us, except perhaps in the example at the end of the paper.

3.4. We find it very useful to make a connection between nilpotent elements in $\mathfrak{g}$ and those in the Lie algebras $\mathfrak{m}_x$ attached to a point $x \in \mathcal{B}_k$. As mentioned in the introduction, these results are related to results of Waldspurger in [Wb]. Recall the Jacobson–Morozov theorem. If N is a nonzero nilpotent element in $\mathfrak{g}$ =Lie$(G)(k)$, then there exists a nontrivial homomorphism ϕ :SL(2) $\longrightarrow G$ (with associated differential $T_\phi : \mathfrak{sl}(2) \longrightarrow \mathfrak{g}$) such that $\phi(e) = N$.

3.5. Proposition. *Let $\phi : SL(2) \longrightarrow G$ be a nontrivial homomorphism defined over k, and let $J = \phi(SL(2, \mathfrak{o}_k))$. Then*

(1) *There exists a minimal parahoric subgroup $\mathcal{P}_x$ $(x \in \mathcal{B}_k)$ containing J.*
(2) *If $\mathcal{P}_x$ and $\mathcal{P}_y$ $(x, y \in \mathcal{B}_k)$ are two minimal parahoric subgroups containing J, then the natural maps from $P_x \cap P_y$ to M_x and M_y are both surjective. In particular, P_x and P_y are associate parahoric subgroups (in the sense of [MPa:5.1]).*

Proof. Consider the action of the compact group $J_K = J \times \mathrm{Gal}(K/k)$ on the building $\mathcal{B}_K$. By Bruhat–Tits theory, the fixed point set $F = \mathcal{B}_K^{J_K}$ (a subset of $\mathcal{B}_k$) is nonempty. The set F is convex, a union of facets and $J \subset \mathcal{P}_x$ for any $x \in F$. Furthermore, $\mathcal{P}_x$ is a minimal parahoric subgroup containing J precisely if the facet determined by x, i.e. the facet of smallest dimension containing x, is maximal, i.e. not contained in a facet of strictly larger dimension. To proceed with the proof of part (2), we introduce some notation and prove a lemma.

Define a linear simplex of dimension r in an Euclidean space E to be the convex closure of $r+1$ points in general position, i.e. the $r+1$ points do not lie in an $(r-1)$–dimensional affine subspace. If S is an r–dimensional linear simplex, let $A(S)$ be the smallest affine subspace containing S. The dimension of $A(S)$ is r.

3.6. Lemma. *Suppose F is a convex subset of an Euclidean space E and suppose F has a (locally finite) cellular decomposition by linear simplicies. If S is a simplex in the cellular decomposition with the property that S is not properly contained in any larger simplex of F, then $F \subset A(S)$.*

Proof of Lemma. Suppose $F \not\subseteq A(S)$. Choose $x \in F - A(S)$. Since F is convex, it contains the convex closure of S and x. This implies S is contained in a (strictly) larger simplex of F, a contradiction. Hence $F \subset A(S)$. $\square$

Suppose S_1 and S_2 are two maximal facets of the convex set $F = \mathcal{B}_K^{J_K}$. Let A be an apartment in $\mathcal{B}_K$ which contains both S_1 and S_2 and consider the convex set F_A which is the intersection of F with A. We note that A, see section 3.1 of [MPb], corresponds to a maximal K-split torus T of G which is defined over k. Let $A(S_1)$ and $A(S_2)$ be the smallest affine subspaces of A containing S_1 and S_2 respectively. By Lemma 3.6, we conclude that i) $A(S_1) = A(S_2)$ (call this common affine subspace A'), ii) that $F_A \subset A'$ and iii) S_1 and

S_2 are open subsets of A'. As in section 6.1 of [MPb], if $z \in A$, then M_z is generated by the image of the maximal compact subgroup T_b of the K–split torus associated to A and the affine root groups U_α such that $\alpha(z) = 0$. Since x and y are in the interiors of maximal facets of A', it it follows that $\alpha(x) = 0$ if and only if $\alpha(y) = 0$ and therefore the natural maps from $P_x \cap P_y$ to M_x and M_y are onto. This proves part (2) of the proposition. $\square$

As an important corollary of Proposition 3.5, we have

3.7. Corollary. *Given a nonzero nilpotent element $N \in \mathfrak{g}$, let $N = e, h, f$ be a $\mathfrak{sl}(2)$ triple associated to N. There exists a point $x \in \mathcal{B}_k$ so that*

1. *$e, h, f \in \mathfrak{g}_{x,0}$, and the images of e, h, f in $\mathfrak{m}_x = \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0^+}$ generate a $\mathfrak{sl}(2)$ triple in $\mathfrak{m}_x$*
2. *Among those parahoric subgroups $P_{x'}$ such that $\mathfrak{g}_{x',0}$ satisfies (1), P_x is minimal.*
3. *If P_y is another parahoric subgroup minimal among those parahoric subgroup for which $\mathfrak{g}_{y,0}$ satisfies (1), then P_x and P_y are associates.*

Corollary 3.7 allows us to associate to any nilpotent orbit in $\mathfrak{g}$ a unique associate class of parahoric subgroups. In the converse direction to Corollary 3.7, we have

3.8. Proposition. *Given an $\mathfrak{sl}(2)$ triple (defined over $\mathbb{F}_q$) $\overline{e}, \overline{h}, \overline{f}$ in $\mathfrak{m}_x$, there is a lift to a $\mathfrak{sl}(2)$ triple e, h, f in $\mathfrak{g}$.*

Proof. We mimic the proof of the Jacobson–Morozov theorem in [Ko]. We will use the following variant of Nakayama's Lemma.

3.9. Lemma. *Suppose V and W are $\mathfrak{o}$–lattices, i.e. free finite rank $\mathfrak{o}$–modules and*

$$A : V \longrightarrow W \qquad \text{is an } \mathfrak{o}\text{–module map}$$

such that the quotient map

$$\overline{A} : V/\wp V \longrightarrow W/\wp W \qquad \text{is surjective.}$$

Then, A is surjective.

We first show that $\overline{e}$ lifts to a nilpotent element $e' \in \mathfrak{g}_{x,0}$. Let G_1 be the 1-parameter (multiplicative) subgroup in M_x so that $\overline{h}$ lies in the Lie algebra Lie(G_1). Lift G_1 to a 1-parameter k-subgroup G_1 of G so that $G_1(\mathfrak{o}_k) = G_1(k) \cap P_x$, and identify the (algebraic) characters of G_1 and G_1. Let $\mathfrak{g}(i)$ be the i-th weight space ($i \in \mathbb{Z}$) for G_1 under Ad. Let $\mathfrak{g}_{x,0}(i) = \mathfrak{g}(i) \cap \mathfrak{g}_{x,0}$ and $\mathfrak{g}_{x,0^+}(i) = \mathfrak{g}(i) \cap \mathfrak{g}_{x,0^+}$. Note that

$$\mathfrak{g}_{x,0} = \bigoplus_i \mathfrak{g}_{x,0}(i)$$

and similarly for $\mathfrak{g}_{x,0^+}$. Choose an $e' \in \mathfrak{g}_{x,0}(2)$ whose image is $\overline{e}$. Note that e is nilpotent. It is the desired nilpotent lift of $\overline{e}$.

Let m be the smallest nonnegative integer so that $(ade')^{m+1} = 0$. An application of Nakayama's lemma shows that $ad(e')^2$ maps $\mathfrak{g}_{x,0}(-2)$ *onto* $\mathfrak{g}_{x,0}(2)$. In particular there is an $f' \in \mathfrak{g}_{x,-2}(0)$ such that $ad(e')^2(f') = e'$. Let $h'' = [f', e']$. By the proof of Lemma 3.4 in [Ko], the ad eigenvalues of h'' on $Cent_{\mathfrak{g}_{x,0}}(e')$ are $i/2$ with $0 \leq i \leq m$. If the characteristic p is large enough, in particular if p satisfies (2.1.1) for all the simple factors of $\mathfrak{m}_x$, then we conclude that $ad(\overline{h''}) + 1$ is nonsingular. By Nakayama's lemma, we conclude that $ad(h'') + 1$ is onto when restricted to $\mathrm{Cent}_{\mathfrak{g}_{x,0}}(e')$. This is Corollary 3.4 in [Ko]. Then the proof of Theorem 3.4 in [Ko] applies and we find the desired triple e, h, f. □

Remark. Propositions 3.5 and 3.8 give a classification of the unipotent orbits of G in terms of associate classes of parahoric subgroups $\mathcal{P}_x$ and unipotent orbits in M_x.

3.10. For an arbitrary element $Y \in \mathfrak{g}_{x,0}$ $(= \mathbf{g}_{x,0} \cap \mathfrak{g})$, denote by $\overline{Y}$ the image in $\mathfrak{m}_x^F$. Consider e, h, f, an $\mathfrak{sl}(2)$ triple as in the first part of Corollary 3.7. Let $\mathfrak{g}(i)$ be the i–th weight space of the element h, and $\mathfrak{g}(\leq i)$ be the sum of the eigenspaces with eigenvalues $\leq i$. Note that any $\mathfrak{g}_{x,r_j}$ is a direct sum of the spaces $\mathfrak{g}_{x,r_j}(i) = \mathfrak{g}(i) \cap \mathfrak{g}_{x,r_j}$. Since we are assuming that p satisfies (2.1.1) for all the simple factors of $\mathfrak{m}_x$, it follows that the representations of $\mathfrak{sl}(2)(\mathbb{F}_q)$ occuring in $\mathfrak{g}_{x,0}/\varpi\mathfrak{g}_{x,0}$ behave as in the characteristic zero theory – an irreducible representation is multiplicity free on restriction to the Cartan subalgebra determined by $\overline{h}$. Also, $\overline{e}$ (*resp.* $\overline{f}$) acts as a raising (*resp.* lowering) operator.

3.11. Lemma. *Suppose that $X = e + z + w$ $(z \in \mathfrak{g}_{x,0}(\leq 0),\ w \in \mathfrak{g}_{x,r_1})$. Then there exists $g \in \mathcal{P}_x$ such that $Ad(g)X = e + z'$ where $z' \in \mathfrak{g}_{x,0}(\leq 0)$ and $z' = z$ mod $\mathfrak{g}_{x,r_1}$.*

Proof. For any index r_j, the quotient $L = \mathfrak{g}_{x,r_j}/\mathfrak{g}_{x,r_{j+1}}$ is a $\mathfrak{g}_x$–module. Let $L(i) = \{x \in L | [\overline{h}, x] = ix\}$ be the i-th weight space of $\overline{h}$ in L. Define $L(\leq 0)$ and $L(\geq 0)$ in the obvious way. A simple calculation shows that given $Y \in L(\geq 0)$, there exists a $U \in L$ such that

$$ad(\overline{e} + \overline{z})(U) = Y \text{ modulo } L(\leq 0). \tag{3.11.1}$$

Claim. If $r_j > 0$, then any element $X = e + z' + w'$ with $z' \in \mathfrak{g}_{x,0}(\leq 0)$ and $w' \in \mathfrak{g}_{x,r_j}(\geq 0)$ can be conjugated by an element $g \in \mathcal{P}_{x,r_j}$ so that $Ad(g)(X) = e + z'' + w''$ with $z'' \in \mathfrak{g}_{x,0}(\leq 0)$ and $w'' \in \mathfrak{g}_{x,r_{j+1}}$.

Indeed, choose $U \in \mathfrak{g}_{x,r_j}$ so that $ad(\overline{e} + \overline{z'})(\overline{U}) = -\overline{w'}$ modulo $\mathfrak{g}_{x,r_j}(\leq 0) + \mathfrak{g}_{x,r_{j+1}}$. It follows that $Ad(\exp(U))(X)$ has the required property stated in the claim. The proof of the lemma now follows from the claim and a standard Hensel's lemma argument. □

3.12. Proposition. *If $X = e + z'$ as in Lemma 3.11 is nilpotent, then $e \in \overline{\mathcal{O}(X)}$.*

Proof. Let $\lambda_X : G_m \longrightarrow G$ be a one parameter subgroup so that $Ad(\lambda_X(t))X = t^2X$. The element h also determines a one parameter subgroup $\lambda : G_m \longrightarrow G$ whose weight spaces are the $\mathfrak{g}(i)$'s. Write $X = e + \sum_{i \leq 0} z_i$ where $z_i \in \mathfrak{g}_{x,0,i}$. We have

$$\begin{aligned} Ad(\lambda(t^{-1})Ad(\lambda_X(t))X) &= Ad(\lambda(t^{-1})t^2X) \\ &= e + \sum_{i \leq 0} t^{2-i} z_i. \end{aligned}$$

As $t \to 0$, we conclude e lies in the closure of the adjoint orbit of X. $\square$

3.13. Corollary. *Suppose $X \in e + \mathfrak{g}_{x,0^+}$ is nilpotent. Then $e \in \overline{\mathcal{O}(X)}$.*

3.14. In this section we work over the field K. We identify the residue field of K with $\overline{\mathbb{F}_q}$. Suppose e, h, f is an $\mathfrak{sl}(2)$ triple as in the first part of Proposition 3.6. Then for an arbitrary element in $z \in \mathfrak{g}_{x,0}$, denote by $\overline{z}$ its image in $\mathfrak{m}_x$. We denote by Σ_x the analogue of (2.5.1), namely,

$$\Sigma_x = \overline{e} + C_{\mathfrak{m}_x}(\overline{f}).$$

Because adh has only integer eigenvalues, the inverse image of Σ_x is contained in

$$\sum_{i \leq 0} \mathfrak{g}_{x,0}(i) + \mathfrak{g}_{x,0^+}.$$

3.15. Lemma. *Consider the map*

$$\psi : \mathsf{M}_x \times \Sigma_x \longrightarrow \mathfrak{m}_x, \quad \psi(g,z) = Ad(g)(z).$$

Then there exists a Zariski open set $\mathcal{U}$ containing $\overline{e}$, which is in the image of ψ.

Proof. The map ψ is smooth with differential $T_{\overline{e}}\psi$ which is onto, and all the varieties in question are nonsingular. $\square$

3.16. Proposition. *Suppose $Y = y + \mathfrak{g}_{x,0^+}$ is a nilpotent coset and that the closure of the adjoint orbit of $\overline{Y}$ in $\mathfrak{m}_x$ contains $\overline{e}$. If $X \in y + \mathfrak{g}_{x,0^+}$ is nilpotent, then $e \in \overline{\mathcal{O}(X)(K)}$ (the closure is taken in the p–adic topology).*

Proof. This is an application of Proposition 3.5. By the assumption, the orbit $\mathcal{O}(\overline{Y})$ must meet every Zariski open set containing $\overline{e}$. By Lemma 3.15, there is a $\overline{g} \in \mathsf{M}_x^F$ and $\overline{z} \in \overline{\Sigma}$ such that $\overline{Y} = \overline{g}(\overline{e} + \overline{z})$. Thus we may replace Y by a conjugate (under P_x) such that $\overline{Y} = \overline{e} + \overline{z}$. By Lemma 3.11, we can find a representative $y \in Y$ of the form $y = e + z$, where $z \in \mathfrak{g}_{x,0}(\leq 0)$. Thus Proposition 3.12 applies and we get the desired result. $\square$

4. The Main Theorem

4.1 An element $u \in G(k)$ (*resp.* $n \in \mathfrak{g}$) is called *topologically unipotent* (*resp. topologically nilpotent*) if u is contained in some $\mathcal{P}_{x,0^+}$, (*resp.* n is contained in some $\mathfrak{g}_{x,0^+}$). In particular, $\lim_{m\to\infty} u^{p^m} = 1$ (*resp.* $\lim_{m\to\infty} ad(n)^{p^m} = 0$). Observe also that if $\tau : G(k) \longrightarrow GL(V(k))$ is a faithful rational representation of $G(k)$ and (3.2.1) is satisfied, then the exponential map from $\mathfrak{gl}(V(k))$ to $\mathrm{GL}(V(k))$ is defined on the topologically nilpotent elements and it (the exponential) maps the topologically nilpotent elements to the topologically unipotent elements of $G(k)$.

4.2. We recall now the main result in [Wa]. Let G be a group of the type Γ as in [Wa:II.1] (essentially a quotient of a product of groups of classical type associated to self–dual lattices). Attached to such a group are positive integers $e(G)$ and $d(G)$ [Wa:II.1], as well as a set of primes $\mathrm{P}(G)$.

4.3. Theorem. *([Wa:I.3]) Suppose that p is a prime number satisfying*

$$p \notin \mathrm{P}(G), \quad p \geq 2d(G)e(G)(2d(G)+1) \quad \text{and} \quad val_k(p) \leq \tfrac{p-1}{4e(G)d(G)}.$$

Let (π, V_π) be an irreducible admissible representation such that $V_\pi^{\mathcal{P}_{x,0^+}} \neq \{0\}$ for some parahoric subgroup $\mathcal{P}_x$. Then, the local character expansion

$$\Theta_\pi(\ \exp X\ ) = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X)$$

is valid for all regular topologically nilpotent elements $X \in \mathfrak{g}$.

4.4. We shall now assume the following conditions on the characteristic p of the field k.

(1) p satisfies the conditions of Theorem 4.3,
(2) p satisfies Lemma 3.3.2 with ρ the adjoint representation,
(3) p satisfies the condition of §2.1 for all M_x.

4.5. Theorem. *Under the hypothesis that the characteristic p of k satisfies 4.4, suppose that $e, h, f \in \mathfrak{g}_{x,0^+}$ is an $\mathfrak{sl}(2)$ triple. Let $f_{x,\mathcal{O}}$ be the character of the generalized Gelfand–Graev representation of $\mathsf{M}_x^F = \mathcal{P}_{x,0}/\mathcal{P}_{x,0^+}$ attached to $\overline{e} \in \mathfrak{m}_x^F$. Then*

(1) *$f_{x,\mathcal{O}}$ is supported on the topologically unipotent set,*
(2) *$\hat{\mu}_{\mathcal{O}'}(f_{x,\mathcal{O}}) = 0$ unless $\mathcal{O}$ lies in the closure $\overline{\mathcal{O}'}$.*
(3) *Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two nilpotent orbits in $\mathfrak{g}$ which belong to the same nilpotent orbit in $\mathbf{g}$.*
   i) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are distinct in $\mathfrak{g}$, then $\hat{\mu}_{\mathcal{O}_1}(f_{x,\mathcal{O}_2}) = 0$.
   ii) If $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ in $\mathfrak{g}$, then $\hat{\mu}_{\mathcal{O}}(f_{x,\mathcal{O}}) \neq 0$.
(4) *For any irreducible smooth admissible representation π,*

$$\Theta_\pi(f_{x,\mathcal{O}}) \;=\; \sum_{\sigma \in \widehat{\mathsf{M}_x^F}} m_\pi(\sigma)\langle f_{x,\mathcal{O}}, \sigma\rangle$$

where $m_\pi(\sigma)$ (resp. $\langle f_{x,\mathcal{O}}, \sigma\rangle$) is the multiplicity of σ in π (resp. $f_{x,\mathcal{O}}$).

In statements (2) and (3), for ease of notation, we have denoted by $f_{x,\mathcal{O}}$, the function on the Lie algebra $\mathfrak{g}$ obtained from $f_{x,\mathcal{O}}$ via the exponential map.

Proof. Let $\Gamma_{\overline{e}}$ denote the generalized Gelfand–Graev character of $\mathcal{P}_{x,0}/\mathcal{P}_{x,0^+}$ attached to $\overline{e} \in \mathfrak{m}_x^F$, so that $f_{x,\mathcal{O}} = \widetilde{\Gamma_{\overline{e}}}$. Statement (1) is part (4) of Proposition 2.4. To prove part (2), we need to show $\mu_{\mathcal{O}'}(\mathrm{FT}_{\mathfrak{g}}(f_{x,\mathcal{O}})) = 0$ unless $\mathcal{O} \subset \overline{\mathcal{O}'}$. By the calculations in section 3.3,

$$\mu_{\mathcal{O}'}(\mathrm{FT}_{\mathfrak{g}}(f_{x,\mathcal{O}})) \;=\; \mathrm{vol}(\mathfrak{g}_{x,0^+}){\sum_{Y}}' \,(\mathrm{FT}_{\mathfrak{m}_x^F}\Gamma_{\overline{e}})(Y)\ \mu_{\mathcal{O}'}(\mathcal{O}' \cap Y) \tag{4.5.1}$$

Note that $\mu_{\mathcal{O}'}(\mathcal{O}' \cap Y) \neq 0$ precisely when the intersection $\mathcal{O}' \cap Y \neq \emptyset$. Suppose $y \in \mathcal{O}' \cap Y$. By Proposition 2.7 (1), $(\mathrm{FT}_{\mathfrak{m}_x^F} \Gamma_{\overline{e}})(Y) \neq 0$ only if $\mathcal{O}_{\mathsf{M}_x}(\overline{e})$ is contained in the closure of $\mathcal{O}_{\mathsf{M}_x}(Y)$. Hence, by Proposition 3.16, the product

$$(\mathrm{FT}_{\mathfrak{g}^F} \Gamma_{\overline{e}})(Y) \mu_{\mathcal{O}'}(\mathcal{O}' \cap Y)$$

is non zero only when $\mathcal{O}$ is contained in the closure of $\mathcal{O}'$. This proves (2). In regards to (3), part i) follows from Proposition 2.7 (2). To show (3ii), observe that the sum in (4.5.1) is over the M_x^F conjugates of $\overline{e}$. There are $[\mathsf{M}_x^F : C_{\mathsf{M}_x^F}(\overline{e})]$ such conjugates. The value of $(\mathrm{FT}_{\mathfrak{g}^F}\Gamma_{\overline{e}})$ at these elements is, by Proposition 2.7 (3), $q^{r(\overline{e})}|C_{\mathsf{M}_x^F}(\overline{e})|$. Whence,

$$\begin{aligned}
\hat{\mu}_{\mathcal{O}}(f_{x,\mathcal{O}}) &= \mu_{\mathcal{O}'}(\mathrm{FT}_{\mathfrak{g}}(f_{x,\mathcal{O}})) \\
&= \mathrm{vol}(\mathfrak{g}_{x,0^+})\ [\mathsf{M}_x^F : C_{\mathsf{M}_x^F}(\overline{e})]\ q^{r(\overline{e})}\ |C_{\mathsf{M}_x^F}(\overline{e})|\ \mu_{\mathcal{O}}(\mathcal{O} \cap e + \mathfrak{g}_{x,0^+}) \\
&= q^{r(\overline{e})}\ \mathrm{vol}(\mathcal{P}_{x,0})\ \mu_{\mathcal{O}}(\mathcal{O} \cap e + \mathfrak{g}_{x,0^+})
\end{aligned}$$

Statement (4) is immediate. $\square$

4.6. Let $\mathcal{O}_1$, $\mathcal{O}_2$, ... , $\mathcal{O}_m$ be the k–rational nilpotent conjugacy classes in $\mathfrak{g}$, and let $\preceq$ be the partial ordering $\mathcal{O} \preceq \mathcal{O}'$ if $\mathcal{O}$ is contained in the closure of $\mathcal{O}'$. Choose $e_i \in \mathcal{O}_i$ and an $\mathfrak{sl}(2)$ triple e_i, h_i, f_i. By Proposition 3.5, we can select a point $x_i \in \mathcal{B}_k$ so that $e_i, h_i, f_i \in \mathfrak{g}_{x_i,0}$.

Let $f_{x_i,\mathcal{O}_i}$ be as in Theorem 4.5. Let

$$\Theta_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi) \widehat{\mu}_{\mathcal{O}} \tag{4.6.1}$$

be the local character expansion of Θ_π. Each of the functions $f_{x_i,\mathcal{O}_i}$ is supported in the topologically unipotent set. They can therefore be used as test functions in the local character expansion (4.6.1). This leads to an invertible triangular linear system with the $c_{\mathcal{O}_i}(\pi)$'s as the unknowns. In particular, $c_{\mathcal{O}_j}(\pi)$ can be obtained by using the test functions $f_{x_i,\mathcal{O}_i}$ with $\mathcal{O}_i \succeq \mathcal{O}_j$.

4.7. Corollary. *The complete local character expansion of a depth 0 representation can be computed via the $f_{x_i,\mathcal{O}_i}$'s.*

5. Wave Front Set

5.1 Assume the characteristic p of k satisfies the conditions of section 4.4. Recall that the Fourier transform $\hat{\mu}_{\mathcal{O}}$ of a nilpotent orbit $\mathcal{O}$ satisfies the homogeneity condition

$$\hat{\mu}_{\mathcal{O}}(t^2X) = |t|^{-\dim\mathcal{O}}\hat{\mu}_{\mathcal{O}}(X) \tag{5.1.1}$$

If x is a point in $\mathcal{B}_k$, $r \in \mathbb{R}$ and ℓ is defined as in [MPa] ($\ell = 1$ if G is split), then $\mathfrak{g}_{x,r+\ell} = \varpi\mathfrak{g}_{x,r}$. For $r > 0$, we have $\mathcal{P}_{x,r}$ is the image of $\mathfrak{g}_{0,r}$ under the exponential map. It will be useful to only consider r equal to some r_i. Let $1_{x,r_i}$ be the characteristic function of $\mathcal{P}_{x,r_i}$. When the context is clear we also let $1_{x,r_i}$ denote the characteristic function of $\mathfrak{g}_{0,r_i}$. If (π, V_π) is an arbitrary admissible representation of $G(k)$, the local character expansion $\Theta_\pi = \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\widehat{\mu}_{\mathcal{O}}$ will be valid in a small neighborhood of the identity and thus for r_i sufficiently large

$$\begin{aligned}\mathrm{vol}(\mathcal{P}_{x,r_i})\ \dim V^{\mathcal{P}_{x,r_i}} &= \Theta_\pi(1_{x,r_i})\\ &= \sum_{\mathcal{O}} c_{\mathcal{O}}(\pi)\ \hat{\mu}_{\mathcal{O}}(1_{x,r_i})\end{aligned}$$

Suppose m is a natural integer. By the homogeneity condition (5.1.1), we have

$$\hat{\mu}_{\mathcal{O}}(1_{x,2m\ell}) = \mathrm{vol}(\mathfrak{g}_{x,2m\ell})\hat{\mu}_{\mathcal{O}}(1_{x,0}).$$

Hence,

$$\dim V^{\mathcal{P}_{x,2m\ell}} = \sum_{\mathcal{O}} q^{m\ \dim\mathcal{O}} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(1_{x,0})$$

In particular, the growth (as a function in m) of the dimension of the spaces $V^{\mathcal{P}_{x,2m\ell}}$ is governed by the orbits $\mathcal{O}$ of maximal dimension whose coefficients $c_{\mathcal{O}}$ are non zero. The set

$$WF(\pi) = \bigcup_{c_\pi(\mathcal{O})\neq 0} \overline{\mathcal{O}}$$

is called the wave front set of π. Thus, the dimension of the wave front set governs the growth of the spaces $V^{P_{x,2m\ell}}$.

Define a nilpotent orbit $\mathcal{O}$ to be a wave front set nilpotent for π if $\mathcal{O}$ is an open subset of $WF(\pi)$. In such a situation, choose $e \in \mathcal{O}$ and $f_{x,\mathcal{O}}$ as in 4.5. If $\mathcal{O}'$ is a strictly larger nilpotent orbit than $\mathcal{O}$, by the definition of $WF(\pi)$, we have $c_\pi(\mathcal{O}') = 0$. Hence,

$$\Theta_\pi(f_{x,\mathcal{O}}) \ = \ c_{\mathcal{O}}(\pi)\ q^{r(\overline{e})}\ \mathrm{vol}(\mathcal{P}_{x,0})\ \mu_{\mathcal{O}}(\mathcal{O}\cap e + \mathfrak{g}_{x,0^+})$$

This means a wave front nilpotent $\mathcal{O}$ forces π to contain a representation σ of $\mathcal{P}_{x,0}/\mathcal{P}_{x,0^+}$ which appears in the generalized Gelfand–Graev representation $\Gamma_{\overline{e}}$.

5.2. Proposition. *The wave front set of a smooth admissible representation π of depth zero can be determined from the wave front sets of the finite field representations.*

6. An Example

We give an example of the technique. Assume $p \neq 2$ and let E be the unramified quadratic extension of k. Let G be the unitary group associated to the hermitian form in three variables given by $2x_{-1}\overline{x_1} + x_0\overline{x_0}$. Let

i) $\mathcal{O}(N_r)$ be the regular nilpotent class,
ii) $\mathcal{O}(N_u)$ and $\mathcal{O}(N_\varpi)$ be the two subregular classes (we choose $\mathcal{O}(N_u)$ so that there exists $N \in \mathcal{O}(N_u)$ whose image in $\mathrm{Lie}(G)(\mathbb{F}_q)$ is a subregular nilpotent),
iii) N_0 the trivial class.

Normalize Haar measure on $G(k)$ so that the measure of an Iwahori subgroup is equal to $\frac{q^4-1}{(q+1)(q^3+1)}$. Normalize the invariant measure on a nilpotent orbit $\mathcal{O}$ so that if $N \in \mathcal{O}$ and Γ_N is the associated generalized Gelfand–Graev representation, then $\mu_{\mathcal{O}}(\Gamma_N) = 1$. With these normalizations, the coefficient, $c_{\mathcal{O}}$, of a wave front set orbit $\mathcal{O}$ is an integer equal to the dimension of generalized Whittaker functionals associated to $\mathcal{O}$ [MW]. If π is a square integrable representation, the coefficient of the trivial orbit in the local character expansion of Θ_π is equal to the negative of the formal degree of π, e.g. with these normalizations, the formal degree of the Steinberg representation is 1 and its local character expansion is $\hat{\mu}_r - \hat{\mu}_0$.

The following table lists the nilpotent orbital integrals for the four associated generalized Gelfand–Graev representations.

	Γ_{N_r}	Γ_{N_u}	Γ_{N_ϖ}	Γ_{N_0}
N_r	1	q	1	q^3+1
N_u	0	1	0	$\frac{(q^3+1)q^2}{(q+1)(q^2+1)}$
N_ϖ	0	0	1	$\frac{(q^3+1)(q-1)}{(q^4-1)}$
N_0	0	0	0	1

Given this table, it is easy to compute the local character expansion of various depth zero representations. As two particularly interesting representation, we have

(1) Let Θ_{10} be the character of the supercuspidal representation induced from the irreducible cuspidal unipotent θ_{10} representation of $G(\mathbb{F}_q)$ inflated to $K = G(\mathfrak{o}_k)$. In particular, the formal degree of Θ_{10} is equal to the degree of θ_{10} divided by the measure of K. The representation θ_{10} does not have a finite field Whittaker model; therefore, by Proposition 5.5.2, neither does Θ_{10} and so the coefficient of $\hat{\mu}_r$ in the local character expansion is 0. We see that

$$\Theta_{10} = \hat{\mu}_u - \frac{q(q^2-1)}{q^4-1}\hat{\mu}_0.$$

(2) Let Θ_{DS} be the character of a non Steinberg discrete series representation π which possesses a nonzero Iwahori fixed vector. The representation π has a unique Whittaker model, i.e. $c_{\mathcal{O}(N_r)} = 1$. Let K' be a maximal compact subgroup of G not conjugate to K and let K'_{0+} be as in [MPb]. The space of K'_{0+}-fixed vectors in π is q dimensional. We have

$$\Theta_{DS} = \hat{\mu}_r - \hat{\mu}_\varpi - \frac{q(q^2-1)}{q^4-1}\hat{\mu}_0.$$

It is known that Θ_{10} and Θ_{DS} form an L–packet, and we have been informed by Kottwitz that $\hat{\mu}_r$, $\hat{\mu}_u - \hat{\mu}_\varpi$ and $\hat{\mu}_0$ are all stable distributions.

Note. After this manuscript was submitted, Waldspurger announced that he had determined the stable combinations of nilpotent orbital integrals for classical groups.

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