# CLASSIFICATION OF ONE K–TYPE REPRESENTATIONS

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## 1. Introduction

Let $G$ be a split reductive p-adic group. In [BM1,BM2], the authors reduce the classification of the irreducible unitary representations of $G$ which have a nonzero Iwahori fixed vector, to similar questions regarding unitary representations of the Iwahori and graded Hecke algebras of $G$. It is therefore natural to investigate the unitary representations of the Iwahori and graded Hecke algebras.

The relation between the Iwahori Hecke algebra and the graded algebra is described in [Ls1]. There, Lusztig begins with a Weyl group $W$ and a reduced simple root system $(\mathcal{X}, \mathcal{Y}, R, \check{R}, \Pi)$ whose reflection group is $W$. The graded Hecke algebra $\mathbb{H}$ associated to the root system is an algebra containing the group algebra $\mathbb{C}W$ and the algebra $\mathcal{S}$ of polynomials on $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{Y}$. As a vector space, $\mathbb{H} = \mathbb{C}W \otimes_{\mathbb{C}} \mathcal{S}$. The algebra $\mathbb{H}$ comes equipped with a $*$ operation. A finite dimensional irreducible representation $(\pi, V_\pi)$ of $\mathbb{H}$ is said to be hermitian (*resp.* unitary) if there is a nondegenerate (*resp.* definite) hermitian form $\langle\ ,\ \rangle$ on $V_\pi$ such that $\langle \pi(h)v_1, v_2\rangle = \langle v_1, \pi(h^*)v_2\rangle$ for all $h \in \mathbb{H}$ and $v_1, v_2 \in V_\pi$.

The purpose of this paper is to investigate the following problem.

**Problem.** *Determine the irreducible representations of $\mathbb{C}W$ which can be extended as hermitian representations to $\mathbb{H}$.*

We call such a representation a one K–type representation. Such representations are clearly unitary.

The relation between the unitary dual of the Hecke algebra and the corresponding graded Hecke algebra is studied in [BM2]. In particular, the most important case is when the elements $x \in \mathcal{X} \subset \mathcal{S}$ act with real eigenvalues. This situation is known as the real infinitesimal character case. In [BM2], the description of the unitary dual is reduced to the description in the case of real infinitesimal character. Thus we shall deal only with real infinitesimal character.

The one K–type representations play a role in the residual spectrum of a reductive algebraic group. For $GL(n)$ they appear as local factors of the automorphic forms constructed in [S] later generalized in [J]. In the work of [T], certain representations $a(n, d)$

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are the building blocks for the unitary dual of $GL(n)$. These are cases of one $\mathsf{K}$–type representations.

The main result of this paper is a classification of one $\mathsf{K}$–type representations for simple root systems. This is obtained as follows. Given $\sigma \in \widehat{W}$ and a linear functional $\omega \in X$ set

$$\sigma(\omega) = \frac{1}{2} \sum_{\beta \in R^+} h_\beta \langle \omega, \check{\beta} \rangle \sigma(t_\beta)$$

where the sum runs over the positive roots and $h_\beta$ is a scalar as in [BM2]. After a review of the $*$ operation in section 2, it is shown that assuming a multiplicity one result (which holds for all $\sigma \in \widehat{W}$ except certain representations of $\mathrm{E}_8$) a necessary and sufficient condition for $\sigma$ to extend to a hermitian representation of $\mathbb{H}$ is that the commutator $[\sigma(\omega), \sigma(\omega')]$ vanish for all $\omega, \omega' \in X$. In sections 3 and 4, the individual cases of the classical and the exceptional Lie types are treated. The results are tabulated in Theorems 3.15 and 3.28 and in the tables of section 4. In section 5, we deal with the remaining cases in $E_8$ which are not amenable to the aforementioned techniques (*c.f.* section 2).

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## 2. The graded Hecke algebra

We recall the definition [Ls1] of the graded Hecke algebra. Let $(\mathcal{X}, \mathcal{Y}, R, \check{R}, \Pi)$ be a reduced simple root system. Let $R^+$ be the positive system corresponding to $\Pi$. For $\alpha \in R$, let $s_\alpha \in GL(\mathcal{X})$ be the reflection

$$s_\alpha(x) = x - \langle x, \check{\alpha} \rangle \alpha.$$

Let $W \subset GL(\mathcal{X})$ be the Weyl group generated by the $s_\alpha$'s, and $\mathbb{C}W$ the group algebra of $W$. When we view $s_\alpha$ as an element in $W$, we shall find it convenient to also denote it as $t_\alpha$. To define a graded Hecke algebra, recall that we need a function $h$ defined on the roots $R$ and taking values in the positive rationals $\mathbb{Q}^+$ so that $h(\alpha) = h(\alpha')$ if $\alpha$ and $\alpha'$ are in the same Weyl orbit. We write $h_\alpha$ for $h(\alpha)$.

Let $\mathcal{S}$ be the symmetric algebra of $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$. The commutative algebra $\mathcal{S}$ is the algebra of polynomials on the vector space ${}^L\mathfrak{a} = \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{Y}$. Let $\mathbb{H}$ be the graded Hecke algebra associated to $h$ as in [Ls1]. It has the algebras $\mathbb{C}W$ and $\mathcal{S}$ as subalgebras. In fact as a $\mathbb{C}$ vector space, $\mathbb{H} = \mathbb{C}W \otimes_{\mathbb{C}} \mathcal{S}$. Given $\alpha \in \Pi$ and $x \in \mathcal{X} \subset \mathcal{S}$, the element $t_\alpha$ commutes past the element $x$ according to the relation

$$t_\alpha \cdot x = s_\alpha(x) \cdot t_\alpha + \langle x, \check{\alpha} \rangle h_\alpha \tag{2.1}$$

Observe that if we multiply the function $h$ by a positive rational number $c$, the graded Hecke algebra obtained from $ch$ is isomorphic to the original graded Hecke algebra. As such, we can and often do assume that the function $h$ takes values in the natural numbers $\mathbb{N}$.

**Theorem 2.2.** *([BM2]) The algebra $\mathbb{H}$ is a $*$ algebra with the $*$ involution given by*

$$\begin{aligned} t^* &= t^{-1} \qquad t \in W \\ \omega^* &= -\omega + \sum_{\beta \in R^+} t_\beta \langle \omega, \check{\beta} \rangle h_\beta \qquad \omega \in \mathcal{X} \end{aligned}$$

We use Theorem 2.2 to give a necessary and sufficient condition for a $\sigma \in \widehat{W}$ to extend to a hermitian representation of $\mathbb{H}$. Let $\langle\ ,\ \rangle$ be a positive definite $W$–invariant hermitian form on $V_\sigma$. Up to a positive scalar multiple $\langle\ ,\ \rangle$ is unique. Given $A \in \mathrm{End}_{\mathbb{C}}(V_\sigma)$, the adjoint of $A$ is the endomorphism $A^* \in \mathrm{End}_{\mathbb{C}}(V_\sigma)$ satisfying $\langle Av_1, v_2 \rangle = \langle v_1, A^* v_2 \rangle$ for all $v_1, v_2 \in V_\sigma$. For $\omega \in \mathcal{X}$, let $p_\omega \in \mathbb{C}W$ be the element

$$p_\omega = \frac{1}{2} \sum_{\beta \in R^+} \langle \omega, \check{\beta} \rangle h_\beta t_\beta$$

and set

$$\sigma(\omega) = \frac{1}{2} \sum_{\beta \in R^+} \langle \omega, \check{\beta} \rangle h_\beta \sigma(t_\beta)$$

so that $\sigma(\omega) = \sigma(p_\omega)$.

**Proposition 2.3.** *Suppose $\sigma$ restricts in a multiplicity free fashion to some parabolic subgroup $W_P \neq W$. Then $\sigma$ extends to a one $K$–type representation of $\mathbb{H}$ if and only if*

$$\sigma([p_\omega, p_{\omega'}]) = 0 \qquad \forall\ \omega, \omega' \in \mathcal{X}.$$

*Proof.* We first establish that

$$\pi(\omega^*) = \pi(\omega)^* \text{ for all } \omega \in \mathcal{X}. \tag{2.4}$$

Suppose $\sigma$ extends to a real hermitian representation $\pi$ of $\mathbb{H}$. Then $\mathcal{S}$ is generated by the $W$–translates of the $x \in \mathcal{X}$ satisfying $w(x) = x$ for all $w \in W_P$. In $\mathbb{H}$, such an $x$ commutes with $W_P$; hence, $\pi(x)$ must commute with $\pi(W_P)$. This and the hypothesis that $\pi_{|W_P}$ is multiplicity free implies $\pi(x)$ preserves each $W_P$–subspace of $V_\sigma$. Hence the action of $\pi(x)$ on $V_\sigma$ can be diagonalized. Since the eigenvalues of $\pi(x)$ are assumed to be real, it follows that $\pi(x)^* = \pi(x)$. Since $s_\alpha(x) = t_\alpha x t_\alpha - \langle x, \check{\beta} \rangle h_\beta$, it follows that $\pi(s_\alpha(x))^* = \pi(s_\alpha(x))$ for every $\alpha \in \Pi$. Thus (2.4) follows. As a consequence,

$$\pi(\omega) = \sigma(\omega) = \frac{1}{2} \sum_{\beta \in R^+} \langle \omega, \check{\beta} \rangle h_\beta\ \sigma(t_\beta) \quad \forall\ \ \omega \in \mathcal{X}.$$

It is obvious now that a necessary condition for $\sigma$ to extend to a real hermitian representation of $\mathbb{H}$ is $[\sigma(\omega), \sigma(\omega')] = 0$. Conversely, if $[\sigma(\omega), \sigma(\omega')] = 0$ for all $\omega, \omega' \in \mathcal{X}$, then $\sigma$ can be extended to a real hermitian representation by letting $\omega$ act as $\sigma(\omega)$. $\square$

The following pairs $(W, W_P)$ have the property that any $\sigma \in \widehat{W}$ restricts to $W_P$ as a multiplicity free representation:

$$\begin{gathered}(W(A_n), W(A_{n-1}))\\(W(B_n), W(B_{n-1}))\\(W(D_n), W(D_{n-1}))\\(W(F_4), W(B_3))\\(W(G_2), W(A_1))\\(W(E_6), W(D_5))\\(W(E_7), W(E_6))\end{gathered}$$

The exceptional cases follow from a case by case inspection of the restriction tables of [A]. The classical cases follow from the Littlewood–Richardson rule for type $A$ and its modifications for the other types.

The pair $(W(E_8), W(E_7))$ also has this property except for the representations $\phi_{6075,14}$, $\phi_{6075,22}$, $\phi_{3240,31}$, $\phi_{3240,9}$, $\phi_{5600,19}$, $\phi_{5600,21}$, $\phi_{4536,23}$, $\phi_{4536,18}$ and $\phi_{7168,17}$. The notation is as in [C]. We call these the sporadic cases. *Except in section 5, we will always assume that $\sigma$ is not one of the sporadic cases.*

**Proposition 2.4.** *Let $\omega_1,\ \omega_2 \in \mathcal{X}$ be linearly independent, and such that they satisfy the following conditions:*

1. *the $W$–translates of $\omega_1$ span $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{X}$ modulo its $W$–invariants,*
2. *The translates of $\omega_2$ under $Stab(\omega_1)$ together with $\omega_1$ span $\mathbb{C} \otimes \mathcal{X}$.*

*Then $\sigma \in \widehat{W}$ extends to a one $\mathsf{K}$–type representation if and only if $[\sigma(\omega_1), \sigma(\omega_2)] = 0$.*

**Proof.** It is enough to check that if $[\sigma(\omega_1), \sigma(\omega_2)] = 0$, then $[\sigma(\omega), \sigma(\omega')] = 0$ for all $\omega,\ \omega' \in \mathcal{X}$. We observe that

$$[\sigma(s_\alpha(\omega)), \sigma(s_\alpha(\omega'))] = t_\alpha[\sigma(\omega), \sigma(\omega')]t_\alpha. \tag{2.5}$$

If $[\sigma(\omega_1), \sigma(\omega_2)] = 0$, then it is the case that $[\sigma(\omega_1), \sigma(t(\omega_2))] = 0$ for every $t \in Stab(\omega_1)$. Therefore

$$[\sigma(\omega_1), \sigma(x)] = 0 \qquad \text{for all } x \in \mathbb{C} \otimes \mathcal{X} \tag{2.6}$$

Using (2.5) again, we find

$$[\sigma(s_\alpha(\omega_1)), \sigma(s_\alpha(\omega_2))] = t_\alpha[\sigma(\omega_1), \sigma(\omega_2)]t_\alpha = 0, \tag{2.7}$$

so $[\sigma(w(\omega_1)), \sigma(w(x))] = 0$ for all $w \in W$, and the claim follows. $\square$

**Example 2.8.** We give a simple illustration with $G_2$ of how we intend to use Proposition 2.4 to classify the one K–type representations. The $G_2$ root system lies in the plane in real 3-space defined by $x_1 + x_2 + x_3 = 0$. The twelve roots are

$$R = \{\pm(\epsilon_i - \epsilon_j), \pm(2\epsilon_i - \epsilon_j - \epsilon_k)\}_{i,j,k \ distinct}$$

As simple root, we choose

$$\alpha = (1, -1, 0) \quad \text{and} \quad \beta = (-1, 2, -1).$$

Take

$$\omega_1 = (1, 0, -1) \quad \text{and} \quad \omega_2 = (1, 1, -2).$$

Then

$$\begin{aligned} \rho_{\omega_1} &= h_\alpha \left(t_{(1,-1,0)} + t_{(0,1,-1)} + 2t_{(1,0,-1)}\right) + h_\beta \left(t_{(2,-1,-1)} + t_{(1,1,-2)}\right) \\ \rho_{\omega_2} &= 3h_\alpha \left(t_{(0,1,-1)} + t_{(1,0,-1)}\right) + h_\beta \left(t_{(-1,2,-1)} + t_{(2,-1,-1)} + 2t_{(1,1,-2)}\right) \end{aligned}$$

Of course $[\sigma(\omega_1), \sigma(\omega_2)] = 0$ for any one dimensional representation $\sigma$ of $W(G_2)$ so they are one K–type representations. In the situation when $\sigma$ is either the reflection representation $\sigma_{ref}$ or the other irreducible two dimensional representation $\sigma_{2'}$ we find

$$\begin{aligned} \sigma_{ref}([p_{\omega_1}, p_{\omega_2}]) \neq 0 &\quad \text{for any } h_\alpha, h_\beta \in \mathbb{N} \\ \sigma_{2'}([p_{\omega_1}, p_{\omega_2}]) = 0 &\quad \text{if and only if } h_\beta = h_\alpha \text{ or } h_\beta = 3h_\alpha \end{aligned}$$

## 3. The Classical Cases

The arguments we present for the diagrams of classical type are different from the exceptional types, even though the two strategies apply equally well. For the classical cases we take advantage of the realizations of the representations in terms of harmonic polynomials and the combinatorics for the symmetric group. The main feature is that in lowest degree there are representatives which look like the *sgn* representations of smaller groups.

We use the standard bases and positive systems for each of the types, A, B, C, D. Denote the system by $(R, \Pi, \check{R}, \check{\Pi})$ and the positive root system by $R^+$.

**Type $A_{n-1}$.**

$$\begin{aligned} R = \check{R} &= \{\epsilon_i - \epsilon_j\}_{i \neq j \leq n}, \\ \Pi = \check{\Pi} &= \{\epsilon_i - \epsilon_{i+1}\}, \\ R^+ &= \{\epsilon_i - \epsilon_j\}_{i < j \leq n}. \end{aligned} \tag{3.1}$$

Most of the time it is more convenient to use the lattices corresponding to $gl(n)$ instead of $sl(n)$. We will do so without special mention.

**Type $B_n$.**

$$
\begin{aligned}
R\ &= \{\pm\epsilon_i \pm \epsilon_j,\ \pm\epsilon_i\}_{i<j\le n}, \quad \check{R}\ =\ \{\pm\epsilon_i \pm \epsilon_j\}_{i<j\le n},\ 2\epsilon_i\}_{i<j\le n},\\
\Pi\ &= \{\epsilon_i - \epsilon_{i+1}, \epsilon_n\}, \quad \check{\Pi}\ =\ \{\epsilon_i - \epsilon_{i+1}, 2\epsilon_n\},\\
R^+\ &= \{\epsilon_i \pm \epsilon_j, \epsilon_i\}_{i<j\le n}.
\end{aligned} \tag{3.2}
$$

**Type $C_n$.**

$$
\begin{aligned}
R\ &= \{\pm\epsilon_i \pm \epsilon_j, 2\epsilon_i\}_{i<j\le n}, \quad \check{R}\ =\ \{\pm\epsilon_i \pm \epsilon_j, \epsilon_i\}_{i<j\le n},\\
\Pi\ &= \{\epsilon_i - \epsilon_{i+1}, 2\epsilon_n\}, \quad \check{\Pi}\ =\ \{\epsilon_i - \epsilon_{i+1}, \epsilon_n\},\\
R^+\ &= \{\epsilon_i \pm \epsilon_j, 2\epsilon_i\}_{i<j}.
\end{aligned} \tag{3.3}
$$

**Type $D_n$.**

$$
\begin{aligned}
R\ =\ \check{R}\ &= \{\pm\epsilon_i \pm \epsilon_j\}_{i<j\le n},\\
\Pi\ =\ \check{\Pi}\ &= \{\epsilon_i - \epsilon_{i+1},\ \epsilon_{n-1} + \epsilon_n\},\\
R^+\ &= \{\epsilon_i \pm \epsilon_j\}_{i<j}.
\end{aligned} \tag{3.4}
$$

We now translate Proposition 2.3 and Proposition 2.4 into coordinates. In all cases we take $\omega_1 = \epsilon_1$ and $\omega_2 = \epsilon_2$. Denote by $p_\omega$ the expression $\sum h_\alpha\langle\omega, \check{\alpha}\rangle t_\alpha$. To establish some shorthand notation set

$$t_{ij} = t_{\epsilon_i - \epsilon_j} \quad t'_{ij} = t_{\epsilon_i + \epsilon_j} \quad t_i = t_{\epsilon_i} = t_{2\epsilon_i}$$

For an element $w \in W$, we write $t_w$ for its representative in $\mathbb{C}W$ and $s_w$ for its action (the reflection representation except in type A) on the vector space $\mathbb{R}^n$ and similarly for its action on $S(\mathbb{R}^n)$ and $S((\mathbb{R}^n)^*)$.

We may as well assume that $h_{\epsilon_i \pm \epsilon_j} = 1$. Then

**Type $A_{n-1}$**

$$p_{\omega_1} = \sum_{i=2}^{n} t_{1i}$$

$$p_{\omega_2} = -t_{12} + \sum_{i=3}^{n} t_{2i}$$

**Type $B_n$**

$$p_{\omega_1} = 2h_{\epsilon_1} t_1 + \sum_{i=2}^{n} (t_{1i} + t'_{1i})$$

$$p_{\omega_2} = 2h_{\epsilon_2} t_2 - t_{12} + t'_{12} + \sum_{i=3}^{n} (t_{2i} + t'_{2i})$$

**Type** $C_n$

$$p_{\omega_1} = h_{2\epsilon_1}t_1 + \sum_{i=2}^{n}(t_{1i} + t'_{1i})$$

$$p_{\omega_2} = h_{2\epsilon_2}t_2 - t_{12} + t'_{12} + \sum_{i=3}^{n}(t_{2i} + t'_{2i})$$

**Type** $D_n$

$$p_{\omega_1} = \sum_{i=2}^{n}(t_{1i} + t'_{1i})$$

$$p_{\omega_2} = -t_{12} + t'_{12} + \sum_{i=3}^{n}(t_{2i} + t'_{2i})$$

Of course $h_{\epsilon_1} = h_{\epsilon_2}$ and similarly $h_{2\epsilon_1} = h_{2\epsilon_2}$.

**Theorem 3.6.** *Write $t = t_{12}$, and $t' = t'_{12}$. The condition $[\sigma(\omega_1), \sigma(\omega_2)] = 0$ on a K–type $\sigma$ is equivalent to*

$$[\sigma(t), \sigma(p_{\omega_1})] = 0 \text{ for type } A,$$
$$[\sigma(t) - \sigma(t'), \sigma(p_{\omega_1})] = 0 \text{ for types } B, C, D.$$

**Proof.** Consider type A. Note that $[t_{ij}, p_1] = 0$ for all $i, j \geq 2$ ($p_1$ is invariant under the symmetric group permuting $\{2, \ldots, n\}$). Thus $[p_1, p_2] = [p_1, t_{12}]$ as claimed. For types $B,\ C,\ D$, the same proof applies, $[p_1, t_{ij}] = 0$ as well as $[p_1, t'_{ij}] = 0$ for all $i, j \geq 2$, and in types $B,\ C$ $[p_1, t_i] = 0$ for all $i$ as well. □

We now describe the realizations of the irreducible representations as in [Ls2]. This uses MacDonald's construction, and is well known in the case of type $A$ when $W$ is the symmetric group. In all cases, given a representation $\sigma$, we write down a harmonic polynomial $\pi_\sigma$ with the property that the canonical action of $W$ on $V_\sigma := span\{w \cdot \pi_\sigma\}_{w \in W}$, provides a realization of $\sigma$. With coordinates as before, let $I \subset \{1, \ldots, n\}$. Then define

$$\pi_{A,I} = \prod_{\substack{i<j \\ i,j \in I}} (\epsilon_i - \epsilon_j),$$

(3.7)
$$\pi_{B,I} = \pi_{C,I} = \prod_{\substack{i<j \\ i,j \in A}} (\epsilon_i^2 - \epsilon_j^2) \cdot \prod \epsilon_i,$$

$$\pi_{D,I} = \prod_{\substack{i<j \\ i,j \in A}} (\epsilon_i^2 - \epsilon_j^2).$$

When $I = \{1, \ldots, n\}$, the polynomial $\pi$ affords the $sgn$ representation of the corresponding Weyl group.

**Type $A_{n-1}$.** The representations of the Weyl group are parametrized by partitions. Let $(n_1, \dots, n_k)$ with $n_i \le n_{i+1}$ be a partition of $n$. In terms of tableaus, the $n_i$ are the sizes of the columns. Fix a partition of $\{1, \dots n\}$ into subsets $I^i$ $(i = 1, \dots, k)$ of sizes $n_i$, and set

$$\pi_{I_1,\dots,I_k} = \prod_i \pi_{A,I_i}. \tag{3.8}$$

This is a harmonic polynomial. The span of these polynomials ranging over all partitions into subsets of sizes $(n_1, \dots, n_k)$ generate an irreducible representation $V_\sigma$ which is by definition the irreducible representation attached to the partition $\sigma$. The action of $W$ is $s_w \pi_{I_1,\dots,I_k} = \pi_{wI_1,\dots,wI_k}$.

**Types B, C, D.** In these cases, $\sigma$ is parametrized by a pair of partitions $(\sigma_L, \sigma_R)$, adding up to $n$. In type $D$, the two partitions $(\sigma_L, \sigma_R)$ and $(\sigma_R, \sigma_L)$ parameterize the same representation which is the restriction of the corresponding representation of type $B(n)$, $C(n)$. When $\sigma_L = \sigma_R$ in which case the restriction splits into two irreducible inequivalent composition factors, $(\sigma_R, \sigma_L)_I$ and $(\sigma_R, \sigma_L)_{II}$.

Using the same conventions as for type $A$, partition $\{1, \dots, n\}$ into subsets $J_s$, $K_r$ of sizes equal to the elements of the partitions $\sigma_L$ and $\sigma_R$ respectively. Then the span of the

$$\pi_{I,J} = \prod_s \pi_{D,I_s} \cdot \prod_r \pi_{B,J_r} \tag{3.9}$$

generate an irreducible representation which we call $V_\sigma$.

The next proposition is the crucial combinatorial tool we will use. Assume we are in type A. Given a polynomial $p \in V_\sigma$, let $p_{ij} := s_{ij}p$.

**Proposition 3.10.** *Assume $\sigma$ corresponds to $(n_1, \dots, n_k)$ and let $I_i$ $(i = 1, \dots, k)$ be a partition into disjoint sets of sizes $n_i$. Suppose that $x \in I_r$, with $n_r \le n_s$ and let $p = \pi_{I_1,\dots,I_k}$. Then*

$$\sum_{j \in I_s} p_{xj} = p.$$

**Proof.** It is enough to consider $\sigma$ corresponding to $(n_1, n_2)$ with $n_1 \le n_2$, $I_1 = \{1, \dots, n_1\}$, $I_2 = \{n_1 + 1, \dots, n_1 + n_2\}$, and $x = 1$.

The sum in the Proposition is

$$\sum_{j \in I_2} s_{1j}(p). \tag{3.11}$$

Then any $s_{kl}$ with $1 < k, l \le n_1$ and $k, l \in I_1$ commutes with the operator $\sum_{j \in I_2} s_{1j}$. Similarly any $s_{kl}$ with $n_1 < k, l \le n_1 + n_2$ commutes with $\sum_{j \in I_2} s_{1j}$. Thus the vector on the left transforms like the *sgn* representation of the subgroup $S_{n_1 - 1} \times S_{n_2}$. We claim that there is only one such vector (up to a scalar) in the representation $\sigma$. We apply Frobenius Reciprocity. By the Littlewood–Richardson rule, the induced module from $Sgn$ of $S_{n_1-1} \times S_{n_2}$ to $S_{n_1+n_2-1}$ is formed of representations corresponding to partitions with columns $(a, b)$ with $b \geq n_2$, $a \leq b$, and $a+b = n_2+n_1-1$. Each occurs with multiplicity 1. The induced module from a representation corresponding to $(a, b)$ of $S_{n_1+n_2-1}$ to $S_{n_1+n_2}$ is formed of representations corresponding to partitions with columns $(1, a, b)$, $(a, b+1)$ (only for $a > b$) and $(a+1, b)$, each occuring with multiplicity 1. The claim now follows from the fact that $b \geq n_2$; the representation with columns $(n_1, n_2)$ occurs exactly once (this is where the fact that $n_1 \leq n_2$ is crucial). Therefore

$$\sum_{j \in I_2} p_{1j} = cp, \tag{3.12}$$

for some constant $c$. It remains to show that $c = 1$. In general, for a partition $(n_1, \ldots, n_k)$ of $n$, write

$$\rho(n_1, \ldots, n_k) := (n_1, \ldots, 1, n_2, \ldots, 1, \ldots, n_k, \ldots, 1). \tag{3.13}$$

We plug $\rho(n_1, n_2)$ into the two sides of (3.10). On the *Left Hand Side,* we get

$$\sum_{j \in I_2} p(s_{1j}\rho(n_1, n_2)) \tag{3.14}$$

Unless $s_{1j}$ fixes $\rho$, the term is 0 because $p$ is evaluated at a point that is singular with respect to a root in $I_1$ or $I_2$. There is only one such $s_{1j}$ namely the one for $j = n_1 + 1$. In this case the value is exactly the *Right Hand Side* of (3.12). $\square$

**Example.** Consider the case when $n_1 = 2$ and $n_2 = 3$. The induced module from the $sgn$ representation of $S_1 \times S_3$ is formed of the representations corresponding to partitions $(1, 1, 3)$, $(2, 3)$ and $(5)$ occuring once and $(1, 4)$ occuring twice.

Take the partition $I_1 = \{1, 2\}$ and $I_2 = \{3, 4, 5\}$. The polynomial $p$ is

$$[(\epsilon_1 - \epsilon_2)] \cdot [(\epsilon_3 - \epsilon_4)(\epsilon_4 - \epsilon_5)(\epsilon_3 - \epsilon_5)].$$

The element $\rho = \rho(2, 3)$ is

$$(2, 1, 3, 2, 1).$$

The polynomial $p$ vanishes on any $s_{1j}\rho$ unless $j = 4$, in which case $s_{14}\rho = \rho$.

On the other hand if we use the same $\sigma$ but $n_1 = 3$ and $n_2 = 2$, the induced module from $sgn$ of $S_2 \times S_2$ is formed of the representations corresponding to the partitions $(1, 2, 2)$, $((1, 1, 3)$, $(5)$ once and $(2, 3)$, $(1, 4)$ twice. Formula (3.12) does not necessarily hold any more.

**Theorem 3.15.** *In type $A$ a representation $\sigma$ of $W(A)$ extends to a representation of $\mathbb{H}$ if and only if $n_1 = \cdots = n_k = d$.*

**Proof.** The relation $\sigma([t,\omega_1]) = 0$ is equivalent to $\sigma(\omega_1) = \sigma(t\omega_1 t)$, which in turn is equivalent to

$$\sum_{i\neq 1} s_{1i}(p) = \sum_{j\neq 2} s_{2j}(p). \tag{3.16}$$

It is also equivalent to

$$\sum_{i\neq 1} t_{1i}(p) = \sum_{j\neq k} t_{mj}(p) \quad \text{for any} \quad m \neq 1. \tag{3.17}$$

Assume first that $n_1 = \cdots = n_k = d$. We check (3.16). With the same notational conventions as before, let $I_1,\dots,I_k$ be *any* partition of $\{1,\dots,n\}$ into subsets of size $d$, and let $p = \pi_{I_1,\dots,I_k}$ be as in (3.8). Note that $s_{1j}p = -p$ if $j$ is in the same set as 1. Using this and applying Proposition 3.10, we find that

$$\sigma(p_{\omega_1})(p) = [-(d-1) + (k-1)]p = (k-d)p. \tag{3.18}$$

Then

$$\begin{aligned} &\sigma(p_{\omega_1})\sigma(t_{12})(p) = \sigma(p_{\omega_1})(p_{12}) = (k-d)p_{12},\\ &\sigma(t_{12})\sigma(p_{\omega_1})(p) = \sigma(t_{12})(k-d)p = (k-d)p_{12} \end{aligned} \tag{3.19}$$

The claim follows.

Now assume that the $n_i$ are not all equal. It is enough to show that (3.17) cannot hold for some choice of $p$. Write the sizes of the partition as

$$\underbrace{n_1,\dots,n_1}_{d_1} < \underbrace{n_2,\dots,n_2}_{d_2} < \dots\ .$$

Assume that $I_1,\dots,I_k$ is the partition where $1,2,\dots n_1 \in I_1$, $n_1+1,\dots 2n_1 \in I_2$ and so on, and let $p$ be the corresponding polynomial. We use (3.17) with $m = n_1 d_1 + 1$. Then by applying Proposition 3.10 to the $p$ corresponding to this partition, we can rewrite (3.17) as

$$(d_1 - n_1 + d_2 + \cdots + d_r)p = (d_2 - n_2 + d_3 + \cdots + d_r)p + \sum_{\substack{i\in I_l,\\ 1\le l\le d_1}} p_{mi}. \tag{3.20}$$

This can be rewritten further as

$$\sum_{\substack{i\in I_l,\\ 1\le l\le d_1}} p_{mi} = (d_1 + n_2 - n_1)p. \tag{3.21}$$

To prove that (3.21) cannot hold, we evaluate at a special element. If we substitute in $\rho(n_1,\dots,n_1,n_2,\dots)$, we see that the the *Right Hand Side* is not zero. But the the *Left Hand Side* is zero. Therefore (3.17) cannot hold. $\square$

We now turn to the cases of type B, C and D. Note first that the relation

$$[tt', p_{\omega_1}] = 0 \tag{3.22}$$

holds. Decompose the space of the representation $\sigma$ as $V = V_- \oplus V_+$ according to the eigenvalues $\pm 1$ of $\sigma(tt')$. If $p$ is the polynomial given by (3.9) for a pair of partitions correponding to $\sigma$, then $p$ is in $V_+$ if both 1, 2 are in sets of the same type, and $p$ is in $V_-$ if they are in opposite types.

The relation in Theorem 3.6 is automatically satisfied on $V_+$ because $\sigma(t) = \sigma(t')$ on this space. On $V_-$ the relation in Theorem 3.6 has the simpler form

$$[\sigma(t), \sigma(p_{\omega_1})]p = 0, \text{ for } p \in V_-. \tag{3.23}$$

**Proposition 3.24.** *Assume that $\sigma_L = \emptyset$ or $\sigma_R = \emptyset$. Then $\sigma$ extends to a hermitian representation of $\mathbb{H}$.*

**Proof.** In these cases, $V_- = 0$. The claim follows from the above discussion. $\square$

Assume that $p$ is given by (3.9) for a partition $\{J_1,\dots,J_s,\dots,K_1,\dots,K_r,\dots\}$ of $\{1,\dots,n\}$ into subsets, corresponding to $\sigma$ attached to $(\sigma_L,\sigma_R)$.

**Lemma 3.25.** *Assume that $1 \in J_1$ Then*

$$[s_{1j} + s'_{1j}]p = \begin{cases} 0 & \textit{if } j \in K_s, \\ 2p_{1j} & \textit{if } j \in J_r,\ r \neq 1. \end{cases}$$

*Assume $2 \in K_1$. Then*

$$[s_{2i} + s'_{2i}]p = \begin{cases} 0 & \textit{if } i \in J_r, \\ 2p_{2i} & \textit{if } i \in K_s,\ s \neq 1. \end{cases}$$

**Proof.** This is just the fact that the degree of an $\epsilon_i$ is even in the case of $\pi_{D,K_1,\dots}$ and odd in the case of $\pi_{B,J_1,\dots}$. $\square$

**Corollary 3.26.** *Let*

$$\delta = \begin{cases} 2h_{\epsilon_1} & \textit{in type } B, \\ h_{2\epsilon_1} & \textit{in type } C, \\ 0 & \textit{in type } D. \end{cases}$$

*Suppose $0 \neq p \in V_-$ is of the form (3.9) with $1 \in J_1$ and $2 \in K_1$. Then the relation $\sigma([t,\omega_1])p = 0$ is equivalent to*

$$\sum_{r\neq 1}\sum_{i\in J_r} p_{1i} - \sum_{s\neq 2}\sum_{j\in K_s} p_{2j} = (\delta + m_1 - n_1)p. \tag{3.27}$$

**Proof.** We rewrite $\sigma(t_{12}\omega_1)p - \sigma(\omega_1 t_{12})p = 0$ as

$$s_{12}[2\delta s_1 + \sum_{i\neq 1}(s_{1i} + s'_{1i})]p - s_{12}[2\delta s_2 + \sum_{j\neq 2}(s_{2j} + s'_{2j})]p = 0.$$

Then we use the fact that $p$ is skew invariant with respect to $s_1$, invariant with respect to $s_2$ and (3.25) to derive the relation. □

**Theorem 3.28.** *Let $\delta$ be as in 3.26. Assume that $\sigma_L \neq \emptyset$ corresponds to the partition $m_1 \leq \cdots \leq m_k$, and $\sigma_R \neq \emptyset$ corresponds to the partition $n_1 \leq \cdots \leq n_l$. Then $\sigma$ extends to a hermitian representation of $\mathbb{H}$ if and only all $m_i = d$, all $n_j = f$ and $k - d = l - f + \delta$.*

**Proof.** Let $p \in V_-$ be as in (3.9). We may as well assume $1 \in J_1$ and $2 \in K_1$. Suppose $p$ satisfies (3.27). In cases $B$, $C$, all the terms in relation (3.27) are divisible by $\prod_r \prod_{i\in J_r} \epsilon_i$. Furthermore, $(\epsilon_i - \epsilon_j)(\epsilon_i + \epsilon_j) = \epsilon_i^2 - \epsilon_j^2$. Divide by the common factor $\prod \epsilon_i$ and replace $\epsilon_i^2$ by $\epsilon_i$. We get a polynomial $\tilde{p}$ of the form $\pi_A$ which must satisfy

$$\sum_{r\neq 1}\sum_{i\in J_r} \tilde{p}_{1i} - \sum_{s\neq 2}\sum_{j\in K_s} \tilde{p}_{2j} = (\delta + m_1 - n_1)\tilde{p}. \tag{3.29}$$

Furthermore $\tilde{p}$ factors as $\tilde{p}_B \cdot \tilde{p}_D$ where $\tilde{p}_B$ is the product over the factors for the $J_r$ and $\tilde{p}_D$ is the product over the factors for the $K_s$. Using the skew invariance with respect to the $J_r$ and $K_s$, we find that

$$\begin{aligned} &\sum_{r\neq 1}\sum_{i\in B^r} \tilde{p}_{B,1i} = c_1\tilde{p}_B, \\ &\sum_{s\neq 1}\sum_{i\in D^s} \tilde{p}_{D,2j} = c_2\tilde{p}_D, \qquad c_1 + c_2 = \delta + m_1 - n_1. \end{aligned} \tag{3.30}$$

The terms in the first sum are all skew for permutations of $J_1$, the terms in the second sum are all skew for permutations of $K_1$. Since the difference is skew for both, all the individual terms are skew. By the result in type A, all the $m_r = d$ and all the $n_s = f$. Then the first part of the proof of theorem 3.15 allows us to compute $c_1, c_2$. The relation $c_1 + c_2$ translates into $k - d = l - f + \delta$.

Now assume that the sizes of the $J_r$ satisfy

$$\underbrace{m_1, \ldots, m_1}_{d_1} < \underbrace{m_2, \ldots, m_2}_{d_2} < \ldots .$$

Then $\tilde{p}_B$ is as in the second part of the proof of theorem 3.15. It is enough to show that

$$\sum_{i\neq m} \tilde{p}_{B,im} \neq c\tilde{p}_B \tag{3.31}$$

for $m = m_1 d_1 + 1$. By plugging in $\rho(m_1, \ldots)$ this can hold if and only if $c = 0$. Thus it is enough to show that the *Left Hand Side* is nonzero. This can be achieved by plugging in $w\rho(m_1, \ldots m_k)$ into the *Left Hand Side* , where $w$ interchanges the $m_1$ in the first entry with the entry $m_2$ in the $m_1 d_1 + 1^{st}$ coordinate. □

## 4. The exceptional cases

**Proposition 4.1.** *Let $\sigma \in \widehat{W}$ and let $\chi_\sigma$ be its character. A necessary and sufficient condition that $[\sigma(\omega_1), \sigma(\omega_2)] = 0$ ($\omega_1, \omega_2 \in \mathcal{X}$) is that*

$$\chi_\sigma([p_{\omega_1}, p_{\omega_2}]^2) \;=\; 0$$

**Proof.** The representation $\sigma$ can be taken to be real orthogonal matrices. Each involution $\sigma(t_\alpha)$ in the definition of $\sigma(\omega)$ is therefore a symmetric matrix. The commutator $[\sigma(\omega_1), \sigma(\omega_2)]$ is a real skew symmetric matrix and so $\sqrt{-1}[\sigma(\omega_1), \sigma(\omega_2)]$ is a Hermitian matrix. Hence it must have real eigenvalues. This means

$$\chi_\sigma([p_{\omega_1}, p_{\omega_2}]^2) \;\leq\; 0,$$

with equality precisely when $\sigma(\omega_1)$ and $\sigma(\omega_2)$ commute. $\square$

We use Propositions 3.1 and 4.1 to determine the one K–type representations of the exceptional types. We begin with types $G_2$ and $F_4$.

**Type $G_2$.** Label the Dynkin diagram as

$$\mathrm{o}_1 \equiv \langle\!\equiv \mathrm{o}_2$$

Let $\omega_1$ and $\omega_2$ be as in (2.6) and take $h_\alpha = 1$ and $h_\beta = h$. To evaluate the character values in Proposition 4.1, we need only compute the terms in the square of the commutator up to conjugacy, i.e. we need to only know the conjugacy classes which appear in the square of the commutator and how many elements of each conjugacy class occurs. One can compute that the element $[p_{\omega_1}, p_{\omega_2}]^2$ has the same character values as the element

$$-(4h^4 + 88h^2 + 36) \;-\; 32(h^3 + 3h)t_\alpha t_\beta \;+\; (4h^4 + 88h^2 + 36)(t_\alpha t_\beta)^2 \;+\; 32(h^3 + 3h)(t_\alpha t_\beta)^3$$

From this we recover the results of example 2.8.

**Type $F_4$.** Label the Dynkin diagram as

$$\mathrm{o}_1 \text{---} \mathrm{o}_2 {=}\rangle{=} \mathrm{o}_3 \text{---} \mathrm{o}_4$$

Let

$$\alpha_1 = \epsilon_2 - \epsilon_3, \quad \alpha_2 = \epsilon_3 - \epsilon_4 \quad \alpha_3 = \epsilon_4, \quad \alpha_1 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$$

Let $\gamma = \epsilon_1$ basis vector to $\check{\alpha}_4$. The stablizer of $\gamma$ is a maximal parabolic subgroup $W_P$ and the weights

$$\omega_1 \;=\; \gamma \quad \text{ and } \quad \omega_2 \;=\; t_4(\omega_1).$$

obviously satisfy the hypothesis of Proposition 3.1. The expression for $p_{\omega_1}$ has 15 terms. The calculation of $[p_{\omega_1}, p_{\omega_2}]^2$, or equivalent for our purposes an element $g$ which has the same trace on any $\sigma \in \widehat{W}$, is best done by machine. For our computations, we used the algebra software GAP [G]. Set

$$\begin{aligned} w_3 &= t_1t_2, \quad w_{3'} = t_3t_4 \\ w_{12} &= t_1t_2t_3t_4 \\ w_6 &= w_{12}^2, \quad w_{6'} = w_6t_1t_2t_4t_3, \quad w_{6''} = w_{12}^3t_{12}t_2 \\ w_2 &= t_1t_3, \quad w_{2'} = t_1t_2t_4t_3t_2t_1t_3t_4 \\ w_4 &= t_2t_3, \quad w_{4'} = w_{12}^3 \end{aligned}$$

Take $h_{\alpha_1} = h$ and $h_{\alpha_2} = 1$. A necessary and sufficient condition for a one K–type representation is that $\chi_\sigma$ vanish on the group element

$$\begin{aligned} g = &- 12(h^4 + 6h^2 + 4)1 \ + \ 480h^2(w_{12} - w_6) \ + \ 24h^4(w_{6'} - w_{3'}) \ - \ 24(h^4 + 4)w_{4'} \\ &+ \ 288h(h^2 + 2)(w_2 - w_4) \ + \ 96(w_{6''} - w_3) \ + \ 36(h^4 + 2h^2 + 4)w_{2'} \end{aligned}$$

We see that $\chi_\sigma(g) = 0$ for any of the four 1-dimensional and four 2-dimensional representations. It is also the case that $\chi_\sigma(g)$ vanishes for the 4-dimensional representation whose character values on the $w's$ are

| 1 | $w_2$ | $w_{2'}$ | $w_3$ | $w_{3'}$ | $w_4$ | $w_{4'}$ | $w_6$ | $w_{6'}$ | $w_{6'''}$ | $w_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 0 | 4 | −2 | −2 | 0 | 4 | 1 | −2 | −2 | 1 |

These nine representations yield one K–type representations for any value of $h$. When $h = 1$, there are five additional one K–type representations with dimensions 4, 4, 6, 8, and 8. The two representations of a given dimension differ by a twist of the *sgn* character. Their values on the $w's$ are

| 1 | $w_2$ | $w_{2'}$ | $w_3$ | $w_{3'}$ | $w_4$ | $w_{4'}$ | $w_6$ | $w_{6'}$ | $w_{6'''}$ | $w_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 0 | 0 | 1 | 1 | −2 | 0 | 2 | −1 | −1 | 0 |
| 6 | 2 | −2 | 0 | 0 | −2 | 2 | 3 | 0 | 0 | −1 |
| 8 | 0 | 0 | −1 | 2 | 0 | 0 | −2 | −2 | 1 | 0 |

When $h = 2$, the representations of dimension 4 and 6 are again one K–type representations. The two representations of dimension 8 no longer yield one K–type representations. Instead, two other representations of dimension 8, which again differ by *sgn*, give one K–type representations. Their values on the $w's$ are

| 1 | $w_2$ | $w_{2'}$ | $w_3$ | $w_{3'}$ | $w_4$ | $w_{4'}$ | $w_6$ | $w_{6'}$ | $w_{6'''}$ | $w_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0 | 0 | 2 | −1 | 0 | 0 | −2 | 1 | −2 | 0 |

Consider now the types $E_6$, $E_7$ and $E_8$. Label the Dynkin diagrams of $E_6$, $E_7$ and $E_8$ as

$$\begin{array}{ccccccccc} \circ_1 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 \\ & & & & | & & & & \\ & & & & \circ_2 & & & & \end{array}$$

$$\begin{array}{ccccccccccc} \circ_1 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 & - & \circ_7 \\ & & & & | & & & & & & \\ & & & & \circ_2 & & & & & & \end{array}$$

$$\begin{array}{ccccccccccccc} \circ_1 & - & \circ_3 & - & \circ_4 & - & \circ_5 & - & \circ_6 & - & \circ_7 & - & \circ_8 \\ & & & & | & & & & & & & & \\ & & & & \circ_2 & & & & & & & & \end{array}$$

Let $\gamma \in R$ be the dual basis vector to $\check{\alpha}_2$, $\check{\alpha}_1$, $\check{\alpha}_8$ respectively. The stabilizer of $\gamma$ is a maximal parabolic subgroup $W_P$. In case $E_6$, $W_P$ is the subgroup obtained from the Dynkin diagram by removing the node $s_2$. In the cases of $E_7$ and $E_8$, $W_P$ is the subgroup obtained from the Dynkin diagram by removing the nodes $s_1$ and $s_8$ respectively. Let $s$ denote the removed node in each of the three cases. Then the weights

$$\omega_1 \;=\; \gamma \quad \text{and} \quad \omega_2 \;=\; s(\omega_1)$$

obviously satisfy the hypothesis of Proposition 3.1. Hence, a necessary and sufficient condition for $\sigma$ to be a one K–type representation is that $[\sigma(\omega_1), \sigma(\omega_2)] = 0$. The vanishing of the commutator is equivalent, by Proposition 4.1, to

$$\chi_\sigma([p_{\omega_1}, p_{\omega_2}]^2) \;=\; 0.$$

In the case of $E_6$ the group algebra elements $p_{\omega_1}$ and $p_{\omega_2}$ each have 21 terms. For $E_7$ and $E_8$, the number of terms increase to 33 and 57 respectively. We again used GAP to perform the tedious calculation of an element $g$ which has the same trace as the element $[p_{\omega_1}, p_{\omega_2}]^2$ on any $\sigma \in \widehat{W}$. We tabulate the results in a table. Set

$$w_2 = s_1 s_2, \quad w_3 = s_1 s_3, \quad w_4 = s_1 s_2 s_3 s_1 s_4 s_2 s_3 s_4 s_5 s_4, \quad w_6 = s_2 s_3 s_4 s_5.$$

| conjugacy class | order | $D_5$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|
| 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $w_3$ | 3 | $-4$ | $-8$ | $-14$ | $-26$ |
| $w_2$ | 2 | 5 | 9 | 15 | 27 |
| $w_4$ | 4 | $-4$ | $-12$ | $-30$ | $-90$ |
| $w_6$ | 6 | 4 | 12 | 30 | 90 |

Then, $\sigma \in \widehat{W}$ is a one $\mathsf{K}$–type representation if and only if the character $\chi_\sigma$ vanishes on the following elements

$$\begin{array}{ll} \text{type } E_6 & -1 - 8w_3 + 9w_2 - 12w_4 + 12w_6 \\ \text{type } E_7 & -1 - 14w_3 + 15w_2 - 30w_4 + 30w_6 \\ \text{type } E_8 & -1 - 26w_3 + 27w_2 - 90w_4 + 90w_6 \end{array}$$

**Type** $E_6$**.** There are 25 irreducible representations of $W$ and 7 are one $\mathsf{K}$–type representations. The character values of these 7 representations on the elements $1, w_3, w_2, w_4, w_6$ are

| $1$ | $w_3$ | $w_2$ | $w_4$ | $w_6$ |
|---|---|---|---|---|
| $1'$ | 1 | 1 | 1 | 1 |
| 10 | $-2$ | 2 | 2 | 0 |
| $15'$ | 0 | 3 | $-1$ | $-2$ |
| $24'$ | 0 | 0 | 0 | 2 |

A $'$ on the dimension indicates that the representation and its twist by the $sgn$ character are distinct representations.

**Type** $E_7$**.** As $W(E_7)$ is the direct product of the group of the subgroup $W(E_7)'$ of elements of even length and the center $\{\pm I\}$. A representation $\sigma \in \widehat{W}$ and its twist by the $sgn$ character are distinct. Either both $\sigma$ and $\sigma \otimes sgn$ are one $\mathsf{K}$–type representations or neither is. There are 30 irreducible representations of $W(E_7)'$. Nine of the representations of $W(E_7)'$ are one $\mathsf{K}$–type representations. The character values of these representations on the elements $1, w_3, w_2, w_4, w_6$ are

| $1$ | $w_3$ | $w_2$ | $w_4$ | $w_6$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 15 | 0 | 3 | $-1$ | $-2$ |
| 21 | 6 | 5 | 1 | 2 |
| 35 | 5 | 7 | $-1$ | $-1$ |
| 70 | $-5$ | 6 | 2 | $-1$ |
| 84 | $-6$ | 4 | 4 | 2 |
| 105 | 0 | $-3$ | $-3$ | 2 |
| 105 | 0 | 9 | $-3$ | $-4$ |
| 210 | $-15$ | 10 | 6 | 1 |

**Type** $E_8$**.** There are 18 one $\mathsf{K}$–type representations. Their character values on the five conjugacy classes in question are

| 1 | $w_3$ | $w_2$ | $w_4$ | $w_6$ |
|---|---|---|---|---|
| $1'$ | 1 | 1 | 1 | 1 |
| $50'$ | 5 | 10 | −2 | −3 |
| $84'$ | 21 | 20 | 4 | 5 |
| $175'$ | −5 | 15 | −1 | −5 |
| $525'$ | 30 | 5 | −7 | 6 |
| $700'$ | −20 | 20 | 0 | −4 |
| $972'$ | 0 | 36 | 0 | 0 |
| 168 | −12 | 8 | 8 | 4 |
| 420 | −30 | 20 | 12 | 2 |
| $448'$ | 16 | 32 | 0 | 0 |

We note that Proposition 4.1 also applies to the classical types. Let $\chi_\sigma$ denote the character of $\sigma \in \widehat{W}$. If we combine Proposition 4.1 and Theorem 3.6, we see that in the classical cases, a necessary and sufficient condition for $\sigma$ to be a one K–type representation is that

$$\begin{aligned} &\chi_\sigma([t, p_{\omega_1}]^2) = 0 \text{ for type } A, \\ &\chi_\sigma([t - t', p_{\omega_1})]^2) = 0 \text{ for types } B, C, D. \end{aligned} \tag{4.3}$$

For type $A_{n-1}$ ($n \geq 3$), if we label the Dynkin diagram as

$$\circ_1 \!-\!\!-\! \circ_2 \!-\!\!-\! \circ_3 \!-\!\!-\! \cdots \!-\!\!-\! \circ_n$$

then this amounts to the character $\chi_\sigma$ vanishing on the group element

$$1 \; - \; (n-3)t_1t_3 \; + \; (n-4)t_1t_2$$

As remarked above, $[t, p_{\omega_1}]^2 \neq 1 \; - \; (n-3)t_1t_3 \; + \; (n-4)t_1t_2$. Rather it is the case that these two group algebra elements have the same character values.

For type $B_n$ ($n \geq 2$), label the Dynkin diagram as

$$\circ_1 \!=\!\langle\!=\! \circ_2 \!-\!\!-\! \circ_3 \!-\!\!-\! \cdots \!-\!\!-\! \circ_n$$

A necessary and sufficient condition for a one K–type representation is that $\chi_\sigma$ vanish on the group element

$$\begin{aligned} &(n+2)\{1 - t_1t_2t_1t_2\} \; + \; 8(n-2)\{t_1t_2 - t_1t_3\} \\ &\qquad + \; 2(n-2)(n-3)\{t_2t_3 - t_2t_4 - t_1t_2t_1t_2t_4t_3 + t_1t_2t_1t_2t_3t_4t_2t_3\} \end{aligned}$$

In type $C_n$ $(n \geq 2)$, with Dynkin diagram

$$\mathrm{o}_1 {=}\rangle{=} \mathrm{o}_2 \text{——} \mathrm{o}_3 \text{——} \cdots \text{——} \mathrm{o}_n$$

a necessary and sufficient condition for a one $\mathsf{K}$–type representation is that $\chi_\sigma$ vanish on the group element

$$\begin{aligned}(n-1)\{1\ &- t_1t_2t_1t_2\} \ + \ 4(n-2)\{t_1t_2 - t_1t_3\} \\ &+ \ 2(n-2)(n-3)\{t_2t_3 - t_2t_4 - t_1t_2t_1t_2t_4t_3 + t_1t_2t_1t_2t_3t_4t_2t_3\}\end{aligned}$$

In type $D_n$ $(n \geq 4)$, with Dynkin diagram

$$\begin{array}{ccccccc} \mathrm{o}_1 & \text{——} & \mathrm{o}_3 & \text{——} & \mathrm{o}_4 & \text{——} \cdots \text{——} & \mathrm{o}_n \\ & & | & & & & \\ & & \mathrm{o}_2 & & & & \end{array}$$

$\sigma$ is a one $\mathsf{K}$–type representation if and only if $\chi_\sigma$ vanishes on

$$\{1 - t_1t_2\} \ + \ 2(n-3)\{t_4t_3 - t_4t_1 - t_4t_3t_1t_2 + t_4t_1t_3t_1t_2t_3\}$$

## 5. The sporadic cases

It is enough to consider $\phi_{6075,14}$, $\phi_{3240,9}$, $\phi_{4536,18}$, $\phi_{5600,19}$, and $\phi_{7168,17}$ because the other representations are obtained by tensoring these with *sgn*. We resort to the classification of representations of the Iwahori Hecke algebra in [KL], more precisely the consequences in [BM1]. Let $G$ be the complex reductive group corresponding to the root datum $(\mathcal{X}, \mathcal{Y}, R, \check{R}, \Pi)$. This is dual to the p–adic group in the introduction. We recall that there is a 1–1 correspondence between equivalence classes of irreducible modules of the Iwahori Hecke algebra $\mathcal{H}$ and $G$–conjugacy classes of $\{(s, u, \phi)\}$ where $s \in G$ is semisimple, $u \in G$ is unipotent such that $sus^{-1} = u^q$ and $\phi$ is an irreducible representation of the component group $A(s, u)$ of the centralizer of $s$ and $u$. More precisely, let $A(u)$ be the component group of the centralizer of $u$. There is a canonical map $\pi : A(s, u) \to A(u)$. Attached to $(s, u)$ there is a *standard module* which we call $X(s, u)$ affording an action of $\mathcal{H} \times A(u)$. Then $A(s, u)$ also acts via the canonical map. Let $\phi \in \widehat{A(s, u)}$. Then the $\phi$–isotypic component $X(s, u, \phi)$ is either zero or else has a unique irreducible submodule $L(s, u, \phi)$. Let $\mathcal{H}_K \subset \mathcal{H}$ be the finite Hecke algebra, which is isomorphic to $\mathbb{C}W$. The module $X(s, u, \phi)$ has the same $\mathbb{C}W$ structure as the module $\sum_\psi H^*(\mathcal{B}_u)^\psi$, where the sum is over the $\psi \in \widehat{A}(u)$ for which $\psi \circ \pi$ contains $\phi$ on $A(s, u)$. The action of $W$ is the *Springer* action on the cohomology of the incidence variety of Borel subgroups of $G$ containing $u$. The irreducible module $L(s, u, \phi)$ is the unique module containing the $W$–module $\sigma_{u,\psi}$ which is also called a lowest K–type of $L(s, u, \phi)$. Note that not all $\psi \in \widehat{A}(u)$ have $\sigma$'s attached to them, so

$X(s,u,\phi)$ and $L(s,u,\phi)$ can be zero. Also an $L(s,u,\phi)$ will contain all the lowest K–types $\sigma_{u,\psi}$ such that $\psi\circ\pi$ contains $\phi$.

Decompose $s=s_0s_u$ such that $s_0\in Stab(u,s)$ and $Ad(s_u)u=u^q$. Then $A(s,u)$ can be identified with the component group of the stabilizer of $u$ in $G(s_0)$. In particular, if $A(u)$ is nontrivial and $A_{G(s_0)}=1$, then $L(s,u,\phi)$ cannot be a one K. We will show that this is the case for the sporadic representations if $s_0\neq 1$. We use the notation in [C]. In the case when $s_0=1$, we have $A(u)=A(s,u)$, $X(s,u,\phi)$ is irreducible, and we use [BS] and [BM1] to exhibit several $W$–types in $H^*(\mathcal{B}_u)^\phi$.

$\phi_{5600,19}$ *and* $\phi_{7168,17}$. These two representations are attached to different $\phi$'s of the nilpotent labeled $E_7(a_5)$. The centralizer (or rather the reductive part of the centralizer which is all that matters for this kind of computation) is a subgroup of type $A_1$. Thus we can conjugate any real $s_0$ into a 1–dimensional torus, and if $s_0\neq 1$, then $G(s_0)$ is a Levi component of type $A_5A_1$. Then $u$ is the principal nilpotent in this subgroup, so $A(s,u)=1$. From [C], $A(u)=S_3$. Thus, there is only one $L(s,u,\phi)$ and it contains both $\phi_{5600,19}$ and $\phi_{7168,17}$. In particular, neither $\phi_{5600,19}$ nor $\phi_{7168,17}$ can be a one K-type representations. It remains to analyze the case $s_0=1$. Then $\phi_{7168,17}$ corresponds to the trivial character of $A(u)$ so by [BM1], $H^*(\mathcal{B}_u)^{triv}$ always contains $sgn$. For $\phi_{5600,19}$, [BS] shows that for the corresponding $\phi$, $H^*(\mathcal{B}_u)^\phi$ is not irreducible.

$\phi_{6075,14}$. This corresponds to the trivial character of $A(u)$ where $u$ is $E_7(a_4)$. In this case $A(u)=S_2$ and the centralizer is of type $A_1$. Arguing as before, if $s_0\neq 1$ then $G(s_0)$ is a Levi subgroup of type $A_6A_1$. Again $u$ is the principal nilpotent so $A(s,u)=1$. We complete the argument as in the previous case.

$\phi_{4536,13}$. This representation corresponds to $\phi=triv$ for $u$ of type $D_5+A_2$. In this case $A(u)=S_2$ and the connected component of the centralizer is a torus. if $s_0\neq 1$ then $G(s_0)$ is of type $D_5A_1$ and $u$ is again the principal nilpotent. The same argument as before applies.

$\phi_{3240,9}$. This representation corresponds to $\phi=triv$ for $u$ of type $D_7(a_1)$. Then $A(u)=S_2$, and the identity component of the centralizer is a torus. Then if $s_0\neq 1$, $G(s_0)$ is of type $D_7$ and $u$ is the principal nilpotent in this subgroup. The argument is as before.

Thus none of the sporadic cases afford a one K-type representation.

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