A UNITARITY CRITERION FOR P-ADIC GROUPS

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1. INTRODUCTION

Let $G = G(\mathbb{F})$ be a split reductive group with connected center over a p–adic field $\mathbb{F}$. In [KL2] (see also [G]), the classification of the irreducible representations with Iwahori fixed vectors is obtained. This is a special case of a more general conjecture of Langlands [Ln] and is achieved as follows. Let $\mathcal{I}$ be an Iwahori subgroup of G and $\mathcal{H} = \mathcal{H}(G//\mathcal{I})$ the algebra of $\mathcal{I}$–biinvariant compactly supported functions on G. The category $\mathcal{C}(\mathcal{I})$ of admissible modules generated by their Iwahori fixed vectors is naturally equivalent to the category $\mathcal{C}(\mathcal{H})$ of finite dimensional $\mathcal{H}$–modules. The equivalence from $\mathcal{C}(\mathcal{I})$ to $\mathcal{C}(\mathcal{H})$ is $V \longmapsto V^{\mathcal{I}}$. Under this equivalence V is hermitian precisely when $V^{\mathcal{I}}$ is hermitian.

A long standing question is whether the unitarity of V can be detected on $V^{\mathcal{I}}$ (see [B,C]). The main result in this paper is the following.

THEOREM 1.1. *An irreducible real hermitian module $V \in \mathcal{C}(\mathcal{I})$ is unitary if and only if $V^{\mathcal{I}} \in \mathcal{C}(\mathcal{H})$ is unitary.*

The proof of Theorem 1.1 relies heavily on the classification of irreducible $\mathcal{H}$–modules in [KL2] and the idea of the signature of a hermitian module in [V2].

The methods employed are valid for arbitrary infinitesimal character. However Corollary 4.8 is false for non–real infinitesimal character. Therefore the assumption that V be real hermitian in Theorem 1.1 is essential. We hope to treat the general case in a subsequent paper using different techniques.

Let $K \supset \mathcal{I}$ be a special maximal compact subgroup of G. As a corollary to our methods, we obtain a characterization of the irreducible $\mathcal{H}$–modules in terms of their $\mathcal{H}_W = \mathcal{H}(K//\mathcal{I})$ structure. This can be thought an analogue of the classification of $(\mathfrak{g}, K)$ modules over $\mathbb{R}$ in terms of their lowest K–types [V1]. This is the content of Theorems 4.6 and 6.3.

Iwahori–Matsumoto have defined an involution $'$ on $\mathcal{C}(\mathcal{H})$ hence $\mathcal{C}(\mathcal{I})$, which preserves hermitian representations. The involution $'$ preserves unitarity on $\mathcal{C}(\mathcal{H})$. As an immediate corollary of Theorem 1.1, we have

THEOREM 1.2. *The involution $'$ on real representations of $\mathcal{C}(\mathcal{I})$ preserves unitarity.*

As an application of Theorem 1.2, we obtain a large class of unitary isolated K–spherical representations. They can be described as follows. Let ${}^LG = {}^LG(\mathbb{C})$ be the complex group dual to G. Every map

$$\phi : SL(2,\mathbb{C}) \longrightarrow {}^LG \tag{1.1}$$

gives rise to a real tempered irreducible representation V_ϕ. In Theorem 8.1 we show that $W_\phi = V'_\phi$ is K–spherical. By Theorem 1.2, the W_ϕ's are unitary. They can be viewed as a special case of the more general conjecture in [A].

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2. BASIC RESULTS

We establish notation and review some basic results. Fix a Borel subgroup $B = A_0N_0$ of G. In the notation of [HC] and [BW], let

$$\mathfrak{a}_0^* = X(A_0) \otimes \mathbb{R} \text{ and } \mathfrak{a}_0 = Hom(X(A_0), \mathbb{R}). \tag{2.1}$$

Let $(,)$ denote a Weyl invariant inner product on the complexification $(\mathfrak{a}_0^*)_c$ of $\mathfrak{a}_0^*$. For $P = MN \supset B$, a parabolic subgroup, let

$$\begin{aligned} &\mathfrak{a}^* = X(M) \otimes \mathbb{R}(\subset \mathfrak{a}_0^*) \ , \ \mathfrak{a} = Hom(X(M), \mathbb{R})(\subset \mathfrak{a}_0) \text{ , and} \\ &H : M \longrightarrow \mathfrak{a} \text{ be the Harish–Chandra map.} \end{aligned} \tag{2.2}$$

If A is a maximal split torus in the center of M, view $X(A)$ as a subset of $\mathfrak{a}^*$. If σ is an admissible representation of M and $\nu \in \mathfrak{a}_c^*$, let $I(P, \sigma, \nu)$ denote the induced representation of right translation on the space of functions

$$\begin{aligned} \{f : G \longrightarrow V_\sigma | f &\text{ locally constant and} \\ & f(mng) = \delta^{1/2}(m)\sigma(m)q^{\nu(H(m))}f(g)\}. \end{aligned} \tag{2.3}$$

We call the representation $I(\nu) = I(P, \sigma, \nu)$ where $P = B$ and σ is the trivial representation of A_0, an unramified principal series and any subquotient of it an $\mathcal{I}$–spherical representation. The following well known theorem justifies the term $\mathcal{I}$–spherical.

THEOREM 2.1. *If σ is a subquotient of $I(\nu)$, then σ has a nonzero Iwahori fixed vector. Conversely, if X is an irreducible admissible representation of G with a nonzero Iwahori fixed vector, then X occurs as a subquotient of some $I(\nu)$.*

Let $\mathcal{C}(\mathcal{I})$ be the category of (equivalence classes of) admissible G-modules whose subquotients are generated by their $\mathcal{I}$–fixed vectors. Let

$$\mathcal{H} = \mathcal{H}(G//\mathcal{I}) = \text{ compactly supported } \mathcal{I}\text{–biinvariant functions on } G. \tag{2.4}$$

This is an algebra under convolution. Moreover, $\mathcal{H}$ acts on the set $V^{\mathcal{I}}$ of $\mathcal{I}$–fixed vectors of any admissible representation (π, V) by

$$\pi(f)v = \int_G f(g)\pi(g)vdg \ (v \in V^{\mathcal{I}}, f \in \mathcal{H}). \tag{2.5}$$

Denote by $\mathcal{C}(\mathcal{H})$ the category of finite dimensional $\mathcal{H}$–modules.

THEOREM 2.2. *The map*

$$V \longmapsto V^{\mathcal{I}}$$

is an equivalence of categories between $\mathcal{C}(\mathcal{I})$ *and* $\mathcal{C}(\mathcal{H})$.

Borel has shown in [B] that the inverse functor is

$$V^{\mathcal{I}} \longmapsto C_c^{\infty}(G/\mathcal{I}) \otimes_{\mathcal{H}} V^{\mathcal{I}}. \tag{2.6}$$

Kazhdan and Lusztig classify the irreducible objects in $\mathcal{C}(\mathcal{H})$.

An admissible representation (π, V) of G is hermitian if there is a hermitian form $<,>$ on V such that

$$< \pi(g)v, w >=< v, \pi(g^{-1})w > (v, w \in V, g \in G). \tag{2.7}$$

If (π^h, V^h) denotes the hermitian dual of (π, V), then (π, V) is hermitian if and only if $(\pi, V) \approx (\pi^h, V^h)$. One can also define the notions of hermitian and unitarity for $\mathcal{H}$. Indeed, $\mathcal{H}$ is a * algebra, with

$$f^*(g) = \overline{f(g^{-1})} \quad (f \in \mathcal{H}, g \in G). \tag{2.8}$$

An $\mathcal{H}$–module E is hermitian if there is a hermitian form $<,>$ on E such that

$$< \pi(f)v, w >=< v, \pi(f^*)w > \quad (v, w \in E, f \in \mathcal{H}). \tag{2.9}$$

In the equivalence of categories between $\mathcal{C}(\mathcal{I})$ and $\mathcal{C}(\mathcal{H})$, it follows from (2.6) that the property of being hermitian is preserved. In one direction, we merely restrict the form on V to $V^{\mathcal{I}}$. In the other direction the form is given by

$$< f_1 \otimes v_1, f_2 \otimes v_2 >_V =< \pi(f_2^* * f_1)v_1, v_2 >_{V^{\mathcal{I}}} . \tag{2.10}$$

Clearly, if V is unitary, then so is $V^{\mathcal{I}}$.

3. STANDARD AND SIMPLE MODULES

We review the Langlands conjecture. Let $W_{\mathbb{F}}$ be the Weil group of $\mathbb{F}$. The basic facts about $W_{\mathbb{F}}$ can be found in [T]. A homomorphism

$$\phi : W_{\mathbb{F}} \times SL(2, \mathbb{C}) \longrightarrow {}^LG \tag{3.1}$$

is admissible if the closure of $\phi(W_{\mathbb{F}})$ is reductive. According to a conjecture of Langlands [Ln], refined later by Langlands, Deligne and Lusztig, to each admissible homomorphism ϕ there should correspond a finite packet $\Pi(\phi)$ of representations of $G(\mathbb{F})$. Furthermore, as one varies over all ${}^LG$ conjugacy classes of ϕ's, the packets $\Pi(\phi)$ should partition the representations of G.

Langlands and Deligne have envisioned a description of those ϕ's for which the representations $\Pi(\phi)$ are $\mathcal{I}$-spherical. Let $I_{\mathbb{F}}$ be the inertia subgroup of $W_{\mathbb{F}}$ so that $W_{\mathbb{F}}/I_{\mathbb{F}} = \mathbb{Z}$. Denote the Frobenius generator by ϖ. Then, conjecturally, the $\mathcal{I}$–spherical packets are parameterized by admissible ϕ's satisfying

$$\phi : W_{\mathbb{F}}/I_{\mathbb{F}} \times SL(2,\mathbb{C}) \longrightarrow {}^LG. \tag{3.2}$$

The condition of admissiblity is that $s = \phi(\varpi)$ be semisimple. Call these ϕ's $I_{\mathbb{F}}$-spherical. This is often reformulated with the aid of the Jacobson-Morozov Theorem. Let

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.3}$$

be the standard triple in the Lie algebra $sl(2,\mathbb{C})$. Set $s = \phi(\varpi)$, a semisimple element and $u = \phi(exp(e))$, a unipotent element. Then, the $\mathcal{I}$–spherical packet parameters are pairs of commuting elements

$$(s,u) : \ s \text{ semisimple, } u \text{ unipotent.} \tag{3.4}$$

Denote by q the order of the residue field of $\mathbb{F}$. The element s is often replaced by

$$\sigma = s \cdot \phi(exp((log(q)/2)h). \tag{3.5}$$

The $\mathcal{I}$–spherical packets are then parametrized by pairs

$$(\sigma,u) : \ \sigma \text{ semisimple, } u \text{ unipotent with } \sigma u \sigma^{-1} = u^q. \tag{3.6}$$

The members of a packet $\Pi(\phi)$ have a conjectural parametrization in terms of a component group. Let $\mathcal{Z}$ be the center of ${}^LG$ and $C(\sigma,u)$ the centralizer of σ and u in ${}^LG$. Then, conjecturally, the members of $\Pi(\phi)$ are parametrized by representations of the component group

$$A(\phi,u) = C(\phi,u)/[C(\phi,u)^0\mathcal{Z}]. \tag{3.7}$$

The data (u,σ,q,ρ), $\rho \in A(\phi,u)$ is called the L–group data of an $\mathcal{I}$–spherical representation. In their proof of the Langlands conjecture, Kazhdan and Lusztig construct a packet of $\mathcal{H}$–modules for each pair (ϕ,u). Their construction is in fact more general. To describe their work we need to review the generic affine Hecke algebra.

Let $B = A_0N_0$ be the Borel subgroup in section 2. Denote by (W,S) the Weyl group of G. Let $\mathcal{P}$ denote the weight lattice of a maximal torus in ${}^LG$. Observe that $\mathcal{P}$ can be viewed as the group $Hom[X(A_0) : \mathbb{Z}]$. There is a canonical action $L \longmapsto w(L),\ (w \in W, L \in \mathcal{P})$ of W on $\mathcal{P}$. For $s \in S$, let $\alpha_s \in Hom[\mathcal{P} : \mathbb{Z}]$ be the coweight corresponding to s. The generic affine Hecke algebra is the abstract algebra $\mathcal{H}(X)$ over $\mathbb{C}[X, X^{-1}]$ with generators $T_s,\ (s \in S),\ \theta_L, (L \in \mathcal{P})$ and the relations

$$\begin{aligned}
&T_s^2 = X + (X-1)T_s \text{ for } s \in S,\\
&T_sT_{s'}T_s \cdots = T_{s'}T_sT_{s'} \ldots \ s \neq s', m \text{ factors on both sides,}\\
&\qquad \text{where } m \text{ is the order of } ss' \text{ in } W,\\
&T_s\theta_{s(L)} = \theta_L T_s - (X-1)\theta_L \text{ when } s \in S, L \in \mathcal{P} \text{ and } \alpha_s(L) = 1,\\
&T_s\theta_{s(L)} = \theta_L T_s \text{ when } s \in S, L \in \mathcal{P} \text{ and } \alpha_s(L) = 0,\\
&\theta_L\theta_{L'} = \theta_{L\otimes L'},\ L, L' \in \mathcal{P},\\
&\theta_{L'} = 1 \text{ when } L \text{ is the trivial element of } \mathcal{P}.
\end{aligned} \tag{3.8}$$

The subalgebra $\mathcal{H}_W(X)$ generated by the T_s's is the generic Hecke algebra of (W, S). Denote by $\mathcal{H}(1)$, the specialization of $\mathcal{H}(X)$ to $X = 1$. It is clear that $\mathcal{H}(1)$ is the group algebra $\mathbb{C}[W\mathcal{P}]$, while $\mathcal{H}_W(1)$ is $\mathbb{C}[W]$. When X is specialized to q, Bernstein has shown

$$\mathcal{H}(q) \text{ and } \mathcal{H} \text{ are naturally isomorphic .} \tag{3.9}$$

We are now ready to explain the work of Kazhdan and Lusztig. For ϕ satisfying (3.2), let $\mathcal{B}_u$ denote the complex variety of Borel subgroups of ${}^LG$ containing $u = exp(e)$ (notation 3.3). Decompose

$$s = \phi(\varpi) = s_e s_h \tag{3.10}$$

into its compact and hyperbolic parts. Let $< s_e >$ (resp. $< s_h >$) denote the smallest diagonalizable algebraic subgroup ${}^LG$ containing s_e (resp. s_h). Define $M \subset {}^LG \times \mathbb{C}^\times$ by

$$M = \{(g,1) | g \in < s_e >\} \times \{(g,1) | g \in < s_h >\} \times \{(exp(yh), e^{2y}) | y \in \mathbb{C}\}. \tag{3.11}$$

The group M acts on $\mathcal{B}_u$ by

$$(g,t) \cdot B = gBg^{-1} \ ((g,t) \in M, B \in \mathcal{B}_u). \tag{3.12}$$

Kazhdan-Lusztig define an action of $\mathcal{H}(X)$ on the M–equivariant K–homology group $K^M(\mathcal{B}_u)$. This K–group is a free $R(M)$–module. Any $m \in M$, determines, by evaluation at m, an algebra homomorphism $\mu_m : R(M) \longrightarrow \mathbb{C}$. Set

$$K^M(\mathcal{B}_u)_m = \mathbb{C} \otimes_{\mu_m} K^M(\mathcal{B}_u) \tag{3.13}$$

where K^M is K–homology as in [KL2], section 1.3.

Let $p_2 :\ {}^LG \times \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ be the projection map to the second factor, and set $t = p_2(m)$. Then, $K^M(\mathcal{B}_u)_m$ is a module for $\mathcal{H}(t)$. The group $A(s,u)$ acts on $K^M(\mathcal{B}_u)$ in a fashion which commutes with $\mathcal{H}(X)$ and specialization. For $\rho \in \hat{A}(u,s)$, define a standard module

$$\mathcal{M}_{u,m,\rho} = Hom_{A(u,s)}[\rho : K^M(\mathcal{B}_u)_m] \tag{3.14}$$

whenever it is nonzero.

THEOREM 3.1. *([KL2] 7.11) Fix $t \in \mathbb{C}^\times$ which is not a root of unity. For each map*

$$\phi : W_{\mathbb{F}}/I_{\mathbb{F}} \times SL(2,\mathbb{C}) \longrightarrow {}^LG$$

let

$$m = (\phi(\varpi exp((log(t)/2)h)), t). \tag{3.15}$$

Then

1. *The standard module $\mathcal{M}_{u,m,\rho}$ has a unique simple quotient $\mathcal{L}_{u,m,\rho}$, which we call the Langlands quotient.*
2. *Every simple module of $\mathcal{H}(t)$ is obtained in this fashion.*

The map ϕ is said to be

$$\begin{aligned} &\text{real if } s_e = 1, \\ &\text{tempered if } s_h = 1, \\ &L^2 \text{ if tempered, and no nontrivial torus T centralizes both } s \text{ and } u. \end{aligned} \tag{3.16}$$

When $s = 1$, the $\rho's$ defining standard modules coincide with those occuring in the Springer representation.

THEOREM 3.2. *([KL2] 8.2, 8.3) Assume $t = q$. Identify $\mathcal{H}(q)$ with $\mathcal{H}$. Let $\pi(\phi, \rho)$ be the representation of G whose Iwahori fixed vectors afford $\mathcal{L}_{u,s,q,\rho}$. Then*

1. *$\pi(\phi, \rho)$ is tempered if and only if ϕ is tempered (here, $\mathcal{M}_{u,s,q,\rho} = \mathcal{L}_{u,s,q,\rho}$),*
2. *$\pi(\phi, \rho)$ is square integrable if and only if ϕ is L^2.*

The packet $\Pi(\phi)$ is defined to be the set

$$\Pi(\phi) = \{\mathcal{L}_{u,s,q,\rho} | \rho \in A(u,s) \text{ and } \mathcal{M}_{u,s,q,\rho} \neq 0\}. \tag{3.17}$$

In the next result we specialize X to 1. We focus on a real packet. Here, the group M is connected. Choose m to be 1. Let $K(\mathcal{B}_u)$ be the K–homology group (over $\mathbb{C}$) of $\mathcal{B}_u$. As an immediate consequence of Proposition 5.11 in [KL2], we have the following fundamental result, important in determining the $\mathcal{H}_W(t)$ structure of $\mathcal{M}_{u,s,t,\rho}$.

THEOREM 3.3. *The natural map $K^M(\mathcal{B}_u)_1 \longrightarrow K(\mathcal{B}_u)$ is an isomorphism.*

A real tempered packet is a ϕ which is trivial on $W_{\mathbb{F}}$. It is thus a homomorphism

$$\phi : SL(2,\mathbb{C}) \longrightarrow {}^LG. \tag{3.18}$$

By the Jacobson-Morozov Theorem, the ${}^LG$ conjugacy classes of ϕ's correspond naturally to unipotent conjugate classes.

We close this section with a review of the center of $\mathcal{H}(X)$. Let $\mathcal{A}$ be the group algebra $\mathbb{C}\mathcal{P}$. The $\mathbb{C}[X, X^{-1}]$–subalgebra of $\mathcal{H}(X)$ generated by the θ_L's is $\mathcal{A}[X, X^{-1}]$. The following result of Bernstein describes the center of $\mathcal{H}(X)$.

THEOREM 3.4. *The center of $\mathcal{H}(X)$ is $\mathcal{A}^W[X, X^{-1}]$.*

The center acts as scalars on any simple module. In analogy with real groups the character by which the center acts is called the infinitesimal character. A character of $\mathcal{A}^W$ is a Weyl orbit in $X(A_0) \otimes_{\mathbb{Z}} \mathbb{C}$. The exponential map

$$exp : X(A_0) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow {}^L G \tag{3.19}$$

maps $X(A_0) \otimes_{\mathbb{Z}} \mathbb{C}$ to a maximal torus. It maps the infinitesimal character of $\mathcal{L}_{u,s,q,\rho}$ multiplied by $log(q)$ to a conjugate of s.

4. $\mathcal{H}_W$ STRUCTURE OF TEMPERED MODULES

In this section, we determine the structure of real tempered modules. To accomplish this, we need to review Springer's work on the representations of W.

Given a unipotent element $u \in {}^L G(\mathbb{C})$, consider the cohomology groups $H^*(\mathcal{B}_u, \mathbb{C})$. For brevity, denote these groups by $H^*(\mathcal{B}_u)$. The centralizer $C(u)$ of u in ${}^L G(\mathbb{C})$ acts by conjugation on $\mathcal{B}_u$; therefore $C(u)$ acts on $H^*(\mathcal{B}_u)$. The action is easily seen to factor to the component group

$$A(u) = C(u)/[C(u)^0 \mathcal{Z}]. \tag{4.1}$$

For $\rho \in \hat{A}(u)$, let $H^*(\mathcal{B}_u)^\rho$ be the ρ–component of $H^*(\mathcal{B}_u)$, i.e.

$$H^*(\mathcal{B}_u)^\rho = Hom_{A(u)}[\rho : H^*(\mathcal{B}_u)]. \tag{4.2}$$

Springer has defined an action of W on $H^*(\mathcal{B}_u)$ and proved the following fundamental result.

THEOREM 4.1.

1. *The actions of W and $A(u)$ commute. Thus, W acts on each $H^*(\mathcal{B}_u)^\rho$, $\rho \in \hat{A}(u)$.*
2. *The natural map*
$$H^*(\mathcal{B}) \longrightarrow H^*(\mathcal{B}_u)$$
induced by the inclusion $\mathcal{B}_u \subset \mathcal{B}$ is W–equivariant.
3. *For $\rho \in \hat{A}(u)$, the representation $\sigma(u, \rho)$ of W on $H^{2d_u}(\mathcal{B}_u)$ ($d_u = dim_{\mathbb{C}}(\mathcal{B}_u)$) is irreducible or zero.*
4. $\sigma(u, 1) \neq 0$.
5. *Every $\sigma \in \widehat{W}$ occurs as some $\sigma(u, \rho)$ with $\mathcal{O}_u$ and ρ uniquely determined.*

The map

$$(u, \rho) \longmapsto \sigma(u, \rho) \tag{4.2}$$

is called the Springer correspondence, and the data (u, ρ) Springer data. The Springer correspondence has been calculated explicitly. Let sgn denote the sign character of W.

PROPOSITION 4.2. *When u is the principal unipotent element, $A(u)$ is trivial and*

$$\sigma(u,1) = sgn.$$

COROLLARY 4.3. *For any unipotent element u, the action of W on $H^0(\mathcal{B}_u)^1$ is the sgn representation.*

Proof. The variety $\mathcal{B}_u$ is connected; hence $H^0(\mathcal{B}_u)^1 = H^0(\mathcal{B}_u) = \mathbb{C}$. The action of W on this one dimensional space now follows from the Proposition 4.2 and Theorem 4.1 part (2).

For suggestive notational convenience, call the representation of W on $H^*(\mathcal{B}_u)^\rho$ a standard module for W. Borho and MacPherson have shown

THEOREM 4.4.

(1) *$Hom_W[\sigma(u,\rho) : H^i(\mathcal{B}_u)] = 0$ if $i \neq 2d_u$.*

(2) *$Hom_W[\sigma(x,\rho) : H^*(\mathcal{B}_u)] = 0$ for $u \notin \overline{\mathcal{O}_x}$.*

Theorem 4.4 says that the standard modules of W form a basis for $R(W)$, and the change of basis matrix between the standard and irreducible modules bases of $R(W)$ is (with a suitable ordering) upper triangular.

Let $H_*(\mathcal{B}_u)$ be the direct sum of the homology groups with coefficients in $\mathbb{C}$. This group is also dual to $H^*(\mathcal{B}_u)$ and therefore carries a natural action of $\mathbb{C}W$ induced by the Springer action in cohomology.

We now define an action of $\mathbb{C}W$ on the standard modules $\mathcal{M}_{u,s,t,\rho}$. Consider the group algebra $\mathbb{Q}[X^{1/2},X^{-1/2}]W$ and the Hecke algebra $\mathcal{H}_{\mathbb{Q}[X^{1/2},X^{-1/2}]}$ over $\mathbb{Q}[X^{1/2},X^{-1/2}]$ defined by generators T_s $(s \in S)$, $\theta_L (L \in \mathcal{P})$ and the relations (3.8).

THEOREM 4.5. *There is a natural map*

$$\begin{aligned} \iota : \mathbb{Q}[X^{1/2},X^{-1/2}]W &\longrightarrow \mathcal{H}_{\mathbb{Q}[X^{1/2},X^{-1/2}]}, \\ \iota(w') = \sum_w f_{w',w} T_w,\ & f_{w',w} \in \mathbb{Q}[X^{1/2},X^{-1/2}] \end{aligned} \tag{4.4}$$

which becomes an isomorphism when $\mathbb{Q}[X^{1/2},X^{-1/2}]$ is extended to $\mathbb{Q}(X^{1/2})$.

If X is specialized to $t \in \mathbb{C}^\times$ which is not a nontrivial root of unity, we obtain an isomorphism

$$\iota_t : \mathbb{C}W \longrightarrow \mathcal{H}_W(t).$$

This allows us to view $\mathcal{M}_{u,s,t,\rho}$ as a $\mathbb{C}W$–module.

We are ready to determine the $\mathcal{H}_W(X)$ structure of real modules. Fix a real $I_{\mathbb{F}}$–spherical map ϕ (notation (3.2)). In this case, the group M of (3.11) is

$$M = \{(g,1) | g \in < s_h >\} \times \{(exp(yh), e^{2y}) | y \in \mathbb{C}\}. \tag{4.5}$$

The algebra $R(M)$ is a Laurent polynomial algebra in several variables.

THEOREM 4.6. *For ϕ real and $m \in M$, $K^M(\mathcal{B}_u)_m$ and $H_*(\mathcal{B}_u)$ are isomorphic as $\mathbb{C}W$ modules.*

This is a rephrasing of Theorem 6.4 in [Ls3]. Its proof will appear in [Ls4].

To give some idea of the proof, we show how $K^M(\mathcal{B}_u)$ and $K(\mathcal{B}_u)_1$ are equivalent as $\mathbb{C}W$ modules.

The K–group $K^M(\mathcal{B}_u)$ is a finite rank free module over $R(M)$. Pick a basis $e_1, e_2, \dots, e_d$ of $K^M(\mathcal{B}_u)$ as an $R(M)$–module. For $s \in S$ and $L \in \mathcal{P}$, write

$$T_s(e_i) = \sum_i f^s_{i,j} e_j, \;\; f^s_{i,j} \in R(M), \tag{4.6a}$$

$$\theta_L(e_i) = \sum_i f^\theta_{i,j} e_j, \;\; f^\theta_{i,j} \in R(M). \tag{4.6b}$$

Let

$$\begin{aligned} \Theta_t :&M \times W \longrightarrow \mathbb{C} \\ &(m, w) \longmapsto trace(w, K^M(\mathcal{B}_u)_m). \end{aligned} \tag{4.7}$$

We conclude from Theorem 4.5 and (4.6) that Θ_m is a Laurent polynomial in several variables. Fix an irreducible representation E of W. Let Θ_E be the character of E. The multiplicity of E in $K^M(\mathcal{B}_u)_m$ is of course

$$d_E(m) = |W|^{-1} \sum \Theta_m(w)\overline{\chi_E(w)}. \tag{4.8}$$

This is a Laurent polynomial, which is integer valued. Thus d_E is constant, indeed equal to $d_E((1,1))$, the multiplicity of E in $H_*(\mathcal{B}_u)$. Hence $K^M(\mathcal{B}_u)_m$ and $K^M(\mathcal{B}_u)_1 \approx K(\mathcal{B}_u)$ are equivalent modules of W.

REMARK. By property (1.3.m2) in [KL2], there is a natural isomorphism

$$C : K(\mathcal{B}_u) \longrightarrow H_*(\mathcal{B}_u) \tag{4.9}$$

which is $A(u)$–equivariant. The difficulty is to show that this map is also W–equivariant. The fact that these are equivalent as $\mathbb{C}W$ modules follows from the proof in [Ls4]. The stronger equivariance statement is not known.

We say more about the real tempered case. Here,

$$M = \{(exp(yh), e^{2y}) | y \in \mathbb{C}\}, \tag{4.10}$$

so $R(M) = \mathbb{C}[X, X^{-1}]$. The groups $A(u,s)$ and $A(u)$ coincide. They act on the K–group $K^M(\mathcal{B}_u)$. For $\rho \in \hat{A}(u)$, set

$$\mathcal{M}_{u,\rho} = Hom_{A(u)}[\rho : K^M(\mathcal{B}_u)]. \tag{4.11}$$

The group $\mathcal{M}_{u,\rho}$ is a finite rank free module over $R(M) = \mathbb{C}[X, X^{-1}]$. When X is specialized to t, the resulting module is $\mathcal{M}_{u,s,t,\rho}$. The above proof applies, to yield

THEOREM 4.7. *For ϕ real tempered and m as in (3.15), $\mathcal{M}_{u,m,\rho}$ and $H_*(\mathcal{B}_u)^\rho$ are equivalent $\mathbb{C}W$–modules.*

COROLLARY 4.8. *As $\mathcal{H}_W(q)$–modules, the real tempered representations of $\mathcal{H}$ are linearly independent.*

Proof. This is an easy consequence of Theorem 4.7 and the remark after Theorem 4.4.

5. SIGNATURE THEOREM

In this section, we establish the p-adic analogue of a fundamental result of Vogan in [V2]. It says the signature character of an $\mathcal{I}$–spherical hermitian module V can be expressed in terms of K–characters of tempered modules. The methods employed are Vogan's ideas transposed to the p-adic setting. We begin by recalling the Langlands classification (see [BW,S]) of representations in terms of tempered representations. In the notation of section 2, fix a Borel subgroup $B = A_0 N_0$ of G and a parabolic subgroup $P = MN \supset B$. Let σ be a tempered representation of M, and $\nu \in \mathfrak{a}^*$ satisfying $(\nu, \alpha) > 0$ for all roots α of A occuring in N. Let $\overline{N}$ denote the unipotent radical opposite to N and $\overline{P} = M\overline{N}$. Then

THEOREM 5.1.

(1) *There is an absolutely convergent integral intertwining operator*

$$\mathcal{L} : I(P, \sigma, \nu) = Ind_P^G(\sigma \otimes \nu) \longrightarrow I(\overline{P}, \sigma, \nu) = Ind_{\overline{P}}^G(\sigma \otimes \nu)$$

$$(\mathcal{L}f)(x) = \int_{\overline{N}} f(nx)dn$$

so that

$$J(\sigma \otimes \nu) = I(P, \sigma, \nu)/ker\mathcal{L}$$

is the unique irreducible quotient of $I(P, \sigma, \nu)$.

(2) *Any irreducible admissible representation* π *of* G *occurs uniquely as some* $J(\sigma \otimes \nu)$. *The parameter* ν *is called the Langlands parameter and denoted by* λ_π.

(3) *If* τ *is an irreducible constituent of* $I(P, \sigma, \nu)$, *then* $\lambda_\tau \leq \nu$ *with equality occuring if and only if* $\tau = J(\sigma \otimes \nu)$.

The irreducible hermitian representations are those $J(\sigma \otimes \nu)$, for which there is an $w \in Weyl(G, A)$ satisfying

$$-\nu = w(\nu) \text{ and } \sigma \approx w(\sigma). \tag{5.1}$$

In this situation, $I(P, \sigma, \nu)$ admits a possibly degenerate hermitian form as follows. Pair the two spaces $I(P, \sigma, \nu)$ and $I(P, \sigma, -\nu)$ via the natural pairing

$$(f_1, f_2) = \int_G < f_1(g), f_2(g) >_\sigma dg \ (f_1 \in I(P, \sigma, \nu), f_2 \in I(P, \sigma, -\nu)). \tag{5.2}$$

Because of (5.1), there is a choice of an isomorphism γ between $I(P, \sigma, -\nu)$ and $I(\overline{P}, \sigma, \nu)$ such that the hermitian form on $I(P, \sigma, \nu)$ is given by

$$< f_1, f_2 >= (f_1, \gamma^{-1}\mathcal{L}f_2). \tag{5.3}$$

The choice of γ depends on the isomorphism $\sigma \approx w(\sigma)$ and not on ν. The radical of $<,>$ is precisely $ker\mathcal{L}$. The quotient form on $J(\sigma \otimes \nu)$ is a nonzero multiple of the unique hermitian form.

Suppose $\pi = J(\sigma \otimes \nu)$ is $\mathcal{I}$–spherical. Let (u, τ, q, ρ) be the L–group data for π. The infinitesimal character of π is

$$\chi_\pi = log_q(\tau). \tag{5.4}$$

Let s_q denote $\phi(exp(log(q)/2)h)$. Write τ as $\tau = s_e s_h s_q$, where s_e and s_h are the elliptic and hyperbolic parts of $\phi(\varpi)$ respectively. Then

$$\nu = log_q(s_h) \text{ and } Re(\chi_\pi) = log_q(s_h s_q). \tag{5.5}$$

Fix a special maximal compact subgroup K containing $\mathcal{I}$. Under the appropriate conventions, the group K is $G(\mathcal{R})$ the $\mathcal{R}$–rational points of G, where $\mathcal{R}$ is the ring of integers in $\mathbb{F}$. The Hecke algebra $\mathcal{H}(K//\mathcal{I})$ is a subalgebra of $\mathcal{H}(G//\mathcal{I})$. In fact,

$$\mathcal{H}(K//\mathcal{I}) \text{ is canonically isomorphic to } \mathcal{H}_W(q). \tag{5.6}$$

For ease of notation, we write $\mathcal{H}_W$ for $\mathcal{H}(K//\mathcal{I})$.

Let (π, V) be an irreducible representation of G with a nonzero hermitian form $<,>$. For each irreducible representation δ of K fix a positive definite hermitian form. Let $V(\delta)$ be the δ-isotypical component of V. The finite dimensional space $F(\delta) = Hom_K[\delta : V(\delta)]$ acquires a nondegenerate form from $<,>$. Denote the dimension and signature of $F(\delta)$ by $m(\delta)$ and $(p(\delta), q(\delta))$ respectively. Trivially, $m(\delta) = p(\delta) + q(\delta)$. Define the formal K–character of V to be

$$\theta_K(V) = \sum_\delta m(\delta)\delta. \tag{5.7}$$

Define the signature character of $<,>$ to be the pair of formal sums

$$(\sum_\delta p(\delta)\delta, \sum_\delta q(\delta)\delta). \tag{5.8}$$

THEOREM 5.2. *Suppose V is an irreducible $\mathcal{I}$–spherical representation of G admitting a nonzero hermitian form $<,>$. There are finitely many irreducible tempered $\mathcal{I}$–spherical modules $V_1, \dots, V_n$ and integers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ so that the signature of $<,>$ is*

$$(a_1\theta_K(V_1) + \cdots + a_n\theta_K(V_n)), (b_1\theta_K(V_1) + \cdots + b_n\theta_K(V_n)).$$

The proof of the signature theorem follows along the same lines as Vogan's proof [V2] in the real case. We need a few preliminary results.

THEOREM 5.3. *The K–character of an $\mathcal{I}$–spherical representation is a linear combination of K–characters of tempered representions.*

Proof. It is enough to prove the assertion for an irreducible $\mathcal{I}$–spherical representation V. Let (u, σ, q, ρ) be the L–group data for V, and let σ_e be the compact part of σ. Denote the set of tempered modules $\mathcal{L}_{u,\theta,q,\rho}$ with $\theta_e = \sigma_e$ by $\mathcal{T}(\sigma_e)$. This is a finite set. Write

V in the Langlands classification as $J(\tau \otimes \nu)$. The K-character of $I(P, \tau, \nu)$ is that of the tempered representation $I(P, \tau, 0)$. This means the assertion is true when $J(\tau \otimes \nu)$ equals $I(P, \tau, \nu)$. Suppose they are not equal. Let π be an irreducible composition factor of $I(P, \tau, \nu)$ different from $J(\tau \otimes \nu)$. Write π in the Langlands classification as $J(\tau' \otimes \lambda_\pi)$, where $\lambda_\pi < \nu$. Denote the infinitesimal characters of V, σ and σ' by χ_V, χ_σ and χ'_σ respectively. It follows from

(1) $\sigma' \in \mathcal{T}(s_e)$
(2) $\lambda_\pi + Re(\chi_{\tau'}) = Re(\chi_V) = \nu + Re(\chi_\tau)$

that the length of $\nu - \lambda_\pi$ is bounded below by a positive number. The theorem follows by an induction on the length of the Langlands parameter.

The next result is Lemma 3.9 in [V2].

LEMMA 5.4. *Suppose V is an $\mathcal{I}$–spherical representation admitting a nondegenerate hermitian form $<,>$. Let*

$$(\sum_\delta p(\delta)\delta, \sum_\delta q(\delta)\delta)$$

be the signature of $<,>$. Then, there exist irreducible modules $Y_1, \dots, Y_n$ and irreducible hermitian modules $X_1, \dots, X_m$ such that (in the Grothendieck group)

$$V = \sum_i X_i + \sum_j (Y_j + Y_j^h).$$

Furthermore, if m_j is the K–character of Y_j and (p_i, q_i) is the signature character of X_i, then

$$p(\delta) = \sum_i p_i(\delta) + \sum_j m_j(\delta),$$

$$q(\delta) = \sum_i q_i(\delta) + \sum_j m_j(\delta).$$

Suppose V is an irreducible hermitian $\mathcal{I}$–spherical representation. Let $J(\sigma \otimes \nu)$ be the Langlands classification realization of V. Since V is hermitian, the standard module $I(P, \sigma, \nu)$ carries the hermitian form (5.3). This standard module is a member of the one parameter family

$$X_t = I(P, \sigma, t\nu) \ (t \in \mathbb{R}). \tag{5.9}$$

The representation, X_0 is tempered while X_1 is the standard representation $I(P, \sigma, \nu)$. Because of the decomposition $G = PK$, the spaces X_t, which are all clearly hermitian, can be realized on the fixed space of functions

$$\begin{aligned} X = \{f : K \longrightarrow V_\sigma | f &\text{ locally constant and} \\ &f(mnk) = \sigma(m) f(g) \ m \in M \cap K, \ n \in N \cap K\}. \end{aligned} \tag{5.10}$$

View $\mathcal{L}_t$ as an operator on X. The function $t \longmapsto \gamma^{-1}\mathcal{L}_t$ can be continued to a meromorphic operator. The operators $\gamma^{-1}\mathcal{L}_t$ can be multiplied by a meromorphic function in t which yields analytic intertwining operators $t \longmapsto \mathcal{A}_t$ on X. From this, we can construct an analytic family of hermitian forms $<,>_t$ on X, namely

$$< f_1, f_2 >_t = (f_1, \mathcal{A}_t f_2), \tag{5.11}$$

with the property

$$rad(<,>_t) = ker(\mathcal{A}_t) = ker(\mathcal{L}_t) \text{ for } t > 0. \tag{5.12}$$

We need to know how the signature changes from 0 to 1. It does not change over intervals where the form $<,>$ is nondegenerate. To determine how the signature changes at a reducibility point we review the Jantzen filtration of X at an arbitrary point t_0. This filtration is the sequence of subspaces

$$X = X^0 \supset X^1 \supset \cdots \supset X^N = \{0\} \tag{5.13}$$

defined as follows. The space X^h is the set of vectors $v \in X$ for which there is a neighborhood U of t_0 and an analytic function

$$f_v : U \longrightarrow X \tag{5.14}$$

satisfying

(1) f_v takes values in a finite dimensional K–subspace of X
(2) $f_v(t_0) = v$
(3) $\forall\ v' \in X$ the function $t \longmapsto < f_v(t), v' >_t$ vanishes at t_0 to order at least n.

Define a hermitian form $<,>^n$ on X^n by the formula

$$< v, v' >^n = \lim_{t \to t_0} \frac{1}{(t - t_0)^n} < f_v(t), f_{v'}(t) >_t, \tag{5.15}$$

where f_v and $f_{v'}$ are choosen as above. The limit depends only on v and v' and not on the particular choice of f_v and $f_{v'}$. Jantzen has shown (see Theorem 3.2 in [V2])

THEOREM 5.5. *The form $<,>^n$ on X^n has radical exactly X^{n+1}.*

Let (p_n, q_n) be the signature character of $<,>^n$. The following theorem is Proposition 3.3 in [V2].

THEOREM 5.6.

(1) *For $t - t_0$ small positive, $<,>_t$ has signature character*

$$(\sum_n p_n, \sum_n q_n).$$

(2) *For $t - t_0$ small negative, $<,>_t$ has signature character*

$$(\sum_{n\ even} p_n + \sum_{n\ odd} q_n, \sum_{n\ odd} p_n + \sum_{n\ even} q_n).$$

Proof of Theorem 5.2. By Theorem 5.3 and Lemma 5.4, we may assume V is an irreducible hermitian representation, say $V = J(\sigma \otimes \nu)$. Let $t_1 < \cdots < t_{r-1}$ be the points in $(0,1)$ where A_t is not an isomorphism. They are the points where $I(P,\sigma,t\nu)$ is reducible. Set $t_0 = 0$ and $t_r = 1$. Let

$$(p^j, q^j) \text{ be the signature of } <,> \text{ on the open interval } (t_{j-1}, t_j). \tag{5.16}$$

Denote the Jantzen filtration of X_{t_j} by

$$X_{t_j} = X^0_{t_j} \supset X^1_{t_j} \supset \dots , \tag{5.17}$$

and the signature of $X^n_{t_j}/X^{n+1}_{t_j}$ by (p^j_n, q^j_n). The representation X_0 is tempered. Consequently $X^n_{t_0}/X^{n+1}_{t_0}$ is tempered and therefore p^0_n and q^0_n are K–characters of tempered representations. By Theorem 5.6, both p^1 and q^1 are linear combinations of tempered K-characters. Exactly as in [V2], we have

$$\begin{aligned} \mathit{signature\ of\ } V = (p^1, q^1) &+ \sum_{l=1}^{r-1} \sum_m (p^l_{2m+1}, q^l_{2m+1}) \\ &- \sum_{l=1}^{r} \sum_m (q^l_{2m+1}, p^l_{2m+1}) - \sum_{m>0} (p^r_{2m}, q^r_{2m}) \end{aligned} \tag{5.18}$$

By induction on the length of the Langlands parameter as explained in Theorem 5.3, we can assume the result for all terms on the right; hence V also satisfies the hypothesis. This completes the proof.
Note that, if V is real then the V_i's can also be taken real.

To be able to apply Theorem 5.2 to the proof of Theorem 1.1, we need to relate the K-character and signature character of a hermitian representation V to analogous characters of $V^{\mathcal{I}}$. Let E be a hermitian $\mathcal{H}$–module. If δ is an irreducible hermitian module of $\mathcal{H}$, let $(p(\delta), q(\delta))$ be the signature of $Hom_{\mathcal{H}_W}[\delta : E]$. In analogy with (5.6) and (5.7), the formal $\mathcal{H}_W$-character and signature character are

$$\theta_{\mathcal{H}_W}(E) = \sum_\delta (p(\delta) + q(\delta))\delta \text{ , and } \quad (\sum_\delta p(\delta)\delta, \sum_\delta q(\delta)\delta) \tag{5.19}$$

respectively. Let $\mathcal{S}$ be the set of irreducible representations of K occuring in the induced representation $Ind^K_{\mathcal{I}} 1$. The map

$$\delta \longmapsto \delta^{\mathcal{I}} \tag{5.20}$$

is a bijection between $\mathcal{S}$ and the simple $\mathcal{H}_W$–modules. Suppose $E = V^{\mathcal{I}}$. Write the K–character of V as the sum

$$\theta_K(V) = \sum_{\delta \in \mathcal{S}} m(\delta)\delta + \sum_{\delta \notin \mathcal{S}} m(\delta)\delta. \tag{5.21}$$

The $\mathcal{H}_W$–character of E is

$$\theta_{\mathcal{H}_W}(E) = \sum_{\delta \in \mathcal{S}} m(\delta)\delta^{\mathcal{I}}. \tag{5.22}$$

The signature character of E is gotten from that of V in the same fashion.

THEOREM 5.7. *Suppose $V_1, \dots, V_m$ are $\mathcal{I}$–spherical real tempered representations of G. Then, a $\mathbb{Z}$ combination*

$$r_1\theta_K(V_1) + \cdots + r_m\theta_K(V_m) = 0$$

if and only if

$$r_1\theta_{\mathcal{H}_W}(V_1^{\mathcal{I}}) + \cdots + r_m\theta_{\mathcal{H}_W}(V_m^{\mathcal{I}}) = 0.$$

Proof This is a consequence of Corollary 4.8.

As an application of Theorems 5.2 and 5.7 we prove Theorem 1.1.

Proof of Theorem 1.1. Denote by $<,>$ the hermitian form on V, and by $<,>^{\mathcal{I}}$ its restriction to $V^{\mathcal{I}}$. Assume $<,>^{\mathcal{I}}$ is positive. We need to show $<,>$ is also positive. Let $\mathcal{T}$ denote the set of $\mathcal{I}$–spherical real tempered representations of G. By Theorem 5.2, the signature of $<,>$ can be written as a $\mathbb{Z}$ combination of tempered K–characters

$$\text{signature of } <,> = \Big(\sum_{\pi\in\mathcal{T}} a_\pi\theta_K(\pi), \sum_{\pi\in\mathcal{T}} b_\pi\theta_K(\pi)\Big). \tag{5.23}$$

Then

$$\text{signature of } <,>^{\mathcal{I}} = \Big(\sum_{\pi\in\mathcal{T}} a_\pi\theta_{\mathcal{H}_W}(\pi), \sum_{\pi\in\mathcal{T}} b_\pi\theta_{\mathcal{H}_W}(\pi)\Big). \tag{5.24}$$

From

(1) the linear independence of the $\theta_{\mathcal{H}_W}(\pi)$'s
(2) and the fact that $<,>^{\mathcal{I}}$ is positive definite,

we conclude the b_π's are zero. Thus $<,>$ is positive.

Iwahori and Matsumoto have defined an algebra involution $'$ on $\mathcal{H}$. It is the specialization of the involution of $\mathcal{H}(X)$ given by

$$T_s \longmapsto -XT_s^{-1} \; s \in S \text{ and } \theta_L \longmapsto \theta_{L^{-1}} \; L \in \mathcal{P}. \tag{5.25}$$

An easy verifcation shows $'$ commutes with the $*$ operation (see 2.7) on $\mathcal{H}$. This means $'$ takes a hermitian module of $\mathcal{H}$ to a hermitian module. The same statement is also true for unitary modules. As a corollary to this observation and Theorem 1.1, we obtain Theorem 1.2.

We observe here that

$$' \text{ preserves the infinitesimal character.} \tag{5.27}$$

This follows from Theorem 3.4 and (5.25).

6. LOWEST K–TYPES

We explain the connection between the standard modules in the Langlands classification and the standard modules of Kazhdan–Lusztig. To do this we need to recall the interpretation of parabolic induction in terms of Hecke algebras. Fix a parabolic subgroup $P = LU$ containing B. The center of L is connected. In particular, the methods of Kazhdan–Lusztig apply to L. We take the Iwahori subgroup of L to be the intersection

$$\mathcal{I}_L = L \cap \mathcal{I}. \tag{6.1}$$

The Hecke algebra $\mathcal{H}_L = \mathcal{H}(L//\mathcal{I}_L)$ can be embedded into $\mathcal{H} = \mathcal{H}(G//\mathcal{I})$ as follows. Denote by S_L the set of simple reflections contained in $Weyl(L, A_0)$. Let $\mathcal{H}_L(X)$ be the $\mathbb{C}[X, X^{-1}]$–subalgebra of $\mathcal{H}(X)$ generated by $\mathcal{A}$ and T_s $s \in S_L$. Let $\mathcal{H}_L(q)$ be the specialization of $\mathcal{H}_L(X)$ to $X = q$. By (3.8), $\mathcal{H}_L = \mathcal{H}_L(q)$. Hence, it is a subalgebra of $\mathcal{H} = \mathcal{H}(q)$. Suppose V is an $\mathcal{I}_L$-spherical representation of L. Bernstein has shown

THEOREM 6.1.

$$(Ind_P^G(V))^{\mathcal{I}} = \mathcal{H} \otimes_{\mathcal{H}_L} V^{\mathcal{I}_L}.$$

Let ${}^L L$ be the Levi subgroup of ${}^L G$ dual to L. Fix an $I_{\mathbb{F}}$-spherical map ϕ

$$\phi : W_{\mathbb{F}}/I_{\mathbb{F}} \times SL(2, \mathbb{C}) \longrightarrow {}^L L. \tag{6.2}$$

Let $A_L(u)$ denote the component group of $u = \phi(exp(e))$ in ${}^L L$, and $\hat{\mathcal{B}}_u$, the variety of Borel subgroups in ${}^L L$ containing u. The K-group $K^M(\hat{\mathcal{B}}_u)_m$ is a module for $\mathcal{H}_L$.

THEOREM 6.2. $([KL2], 7.8)$ *As $\mathcal{H}$-modules*

$$K^M(\mathcal{B}_u)_m = \mathcal{H} \otimes_{\mathcal{H}_L} K^M(\hat{\mathcal{B}}_u)_m.$$

We are ready to compare the standard modules of Langlands and those of Kazhdan-Lusztig. Fix an admissible homomorphism ϕ, and let $\phi(\varpi) = s_e s_h$ be the decomposition (3.10). The centralizer ${}^L L$ of s_h in ${}^L G$ is a Levi subgroup. Define

$$\phi_{temp} : W_{\mathbb{F}}/I_{\mathbb{F}} \times SL(2, \mathbb{C}) \longrightarrow {}^L L \tag{6.3}$$

by $\phi_{temp}(g) = \phi(g)$, $(g \in SL(2, \mathbb{C}))$ and $\phi_{temp}(\varpi) = s_e$. The representation τ of L corresponding to a nonzero $K^M(\hat{\mathcal{B}}_u)_{m,\rho}$ is tempered. The element $\nu = log_q(s_h)$ determines a positive chamber in $\mathfrak{a}^*$. Let $P = LU$ be the parabolic subgroup determined by this chamber. Then

$$\mathcal{M}_{u,m,\rho} = I(P, \tau, \nu)^{\mathcal{I}} \text{ and } \mathcal{L}_{u,m,\rho} = J(\tau \otimes \nu)^{\mathcal{I}}. \tag{6.4}$$

We describe how to determine the simple quotient of a standard module in terms of K–types. Our description applies to the Kazhdan–Lusztig standard modules. This description can be transfered to the Langlands standard modules via (6.4) and (5.20).

Define a partial ordering on the representations of W by

$$\sigma(u,\rho) \preceq \sigma(u',\rho') \text{ when } u \in \overline{\mathcal{O}_{u'}}. \tag{6.5}$$

In this partial ordering, the sgn (resp. trivial) representation of W is a maximal (resp. minimal) element. Fix a ϕ which is real. The group $A(u,s)$ is naturally a subgroup of $A(u)$ because in a connected algebraic group, the centralizer of a torus is connected. For each $\rho \in \hat{A}(u,s)$ we define a subset $\Lambda(\phi,\rho)$ of $\hat{W}$ by

$$\Lambda(\phi,\rho) = \{\sigma(u,\psi) \mid \psi \textit{ a representation of } A(u) \textit{ containing } \rho\}. \tag{6.6}$$

In the special case when ϕ is real tempered, $A(u) = A(u,s)$ and $\Lambda(\phi,\rho) = \{\sigma(u,\rho)\}$.

THEOREM 6.3. *Fix ϕ real and $\rho \in \hat{A}(u,s)$.*

(1) *Each $\sigma \in \Lambda(\phi,\rho)$ occurs in $\mathcal{M}_{u,m,\rho}$. Among the representations of W occuring $\mathcal{M}_{u,m,\rho}$, they are minimal for the partial order $\preceq$.*
(2) *For any $\sigma \in \Lambda(\phi,\rho)$, the smallest subquotient of $\mathcal{M}_{u,m,\rho}$ containing $\sigma(u,\psi)$ is $\mathcal{L}_{u,m,\rho}$.*

Proof. Assertion (1) is a consequence of Theorems 4.4 and 4.2. To prove (2), we do an induction on the dimension of $\mathcal{B}_u$. When u is a principal unipotent element the assertion is obvious. Assume the assertion is true for all u' satisfying $dim(\mathcal{B}_{u'}) < dim(\mathcal{B}_u)$. By Theorem 4.2 and (1), a necessary condition for $\mathcal{L}_{u',m',\rho'}$ to appear as a constituent of $\mathcal{M}_{u,m,\rho}$ is

$$u \in \overline{\mathcal{O}_{u'}}. \tag{6.7}$$

For those u' satisfying (6.9) and $\mathcal{O}_{u'} \neq \mathcal{O}_u$, the $\mathbb{C}W$-module $\mathcal{L}_{u',m',\rho'}$ does not contain any representation in $\Lambda(\phi,\rho)$. The assertion follows.

Thereom 6.3 was obtained earlier for $GL(n)$ by Rogawski [R].

7. REDUCIBILITY

We state here explicitly some results on reducibility of tempered representations which can be found implicity in [KL2]. Suppose (u,σ,q,ρ) are L–group data for a real module. Assume further that the associated map ϕ maps $SL(2,\mathbb{C})$ into a Levi subgroup ${}^LL$ of ${}^LG$. By Theorems 4.1 and 6.1,

$$\begin{aligned}
\oplus_{\rho\in\hat{A}(u)}(K^M(\mathcal{B}_u)_{m,\rho}\otimes\rho) &= K^M(\mathcal{B}_u)_m \\
&= \mathcal{H}\otimes_{\mathcal{H}_L} K^M(\hat{\mathcal{B}}_u)_m \\
&= \mathcal{H}\otimes_{\mathcal{H}_L} (\oplus_{\psi\in\hat{A}_L(u)}K^M(\hat{\mathcal{B}}_u)_{m,\psi}\otimes\psi) \\
&= \oplus_{\psi\in\hat{A}_L(u)}(\mathcal{H}\otimes_{\mathcal{H}_L} K^M(\hat{\mathcal{B}}_u)_{m,\psi})\otimes\psi.
\end{aligned} \tag{7.1}$$

This enables us to easily calculate the multiplicity of the irreducible tempered module $K^M(\mathcal{B}_u)_{m,\rho}$ in the module induced from $K^M(\hat{\mathcal{B}}_u)_{m,\psi}$. Indeed, if $m(\rho,\psi)$ is the multiplicity of ψ in the restriction of ρ to $A_L(u)$, then

$$\mathcal{H}\otimes_{\mathcal{H}_L} K^M(\hat{\mathcal{B}}_u)_{m,\psi} = \oplus_{\rho\in\hat{A}(u)} dim(\rho) m(\rho,\psi) K^M(\mathcal{B}_u)_{m,\rho}. \tag{7.2}$$

We give two examples where the induced module is not multiplicity free.

Example 7.1. Take ${}^LG = E_r(\mathbb{C})$, with $r = 6, 7$ or 8. There is a Levi subgroup ${}^LL$ of type $D_4(\mathbb{C}) \times (\mathbb{C}^\times)^{r-4}$. This group is unique up to conjugacy. Let u be a subregular unipotent element in ${}^LL$. The component group $A_L(u)$ (resp. $A(u)$) in ${}^LL$ (resp. ${}^LG$) is trivial (resp. S_3). The former means

$$K^M(\hat{\mathcal{B}}_u)_m \text{ is irreducible tempered,} \tag{7.3}$$

while the latter means

$$\mathcal{H}\otimes_{\mathcal{H}_L} K^M(\hat{\mathcal{B}}_u)_m = \oplus_{\rho\in\hat{S_3}}(K^M(\mathcal{B}_u)_{m,\rho}\otimes\rho). \tag{7.4}$$

In particular, the multiplicity of $K^M(\mathcal{B}_u)_{m,\theta}$, θ the reflection representation of S_3, in the induced representation is 2.

Example 7.2. Take ${}^LG = E_8(\mathbb{C})$. There is an unique conjugacy class of Levi subgroups of type $D_4(\mathbb{C}) \times A_2(\mathbb{C})$. Fix such an ${}^LL$. Let u be an unipotent element in ${}^LL$ which projects to a subregular (resp. principal) unipotent element in $D_4(\mathbb{C})$ (resp. $A_2(\mathbb{C})$). As in Example 7.1, $A_L(u)$ is trivial, while $A(u)$ is S_3. Formulas (7.3) and (7.4) hold, so the induced representation is again not multiplicity free.

If ${}^LG$ is simple, the above two examples give all instances in which an induced tempered representation is not multiplicity free.

8. UNITARY K-SPHERICAL REPRESENTATIONS

The trivial and sgn represntations of $\mathcal{H}_W$ are the one dimensional representations given on the generators T_s $(s\in S)$ by

$$T_s \longmapsto q, \text{ and } T_s \longmapsto -1 \tag{8.1}$$

respectively. An $\mathcal{I}$-spherical module V is K-spherical precisely when $V^{\mathcal{I}}$ contains the trivial representation of $\mathcal{H}_W$. The Iwahori-Matsumoto involution $'$ of $\mathcal{H}$ restricts to an algebra involution of $\mathcal{H}_W$. The trivial and sgn representation are exchanged by $'$. More generally, under the Lusztig isomorphism between $\mathcal{H}_W$ and $\mathbb{C}W$, the trivial and sgn representation in (8.1) correspond respectively to the trivial and sgn representation of W. The deformation arguments in section 4 can be applied to show that the effect of $'$ on a representation τ of W is

$$\tau \longmapsto \tau\otimes sgn. \tag{8.2}$$

Suppose (u,σ,q,ρ) is L–group data for a real tempered module $\mathcal{M}_{u,\sigma,q,\rho}$. We know from Theorem 4.7 that the action of $\mathbb{C}W$ on $\mathcal{M}_{u,\sigma,q,\rho}$ and $H^*(\mathcal{B})^\rho$ are equivalent. Consider the case when ρ is trivial. By Corollary 4.3, sgn occurs in $H^*(\mathcal{B})^1$. This means $\mathcal{M}_{u,\sigma,q,1}$ contains the sgn represention of $\mathcal{H}_W$. Apply the involution $'$. We conclude

THEOREM 8.1. *The module* $\mathcal{M}'_{u,\sigma,q,1}$ *is*

(1) *K–spherical,*
(2) *unitary and*
(3) *has the same infinitesimal character as the tempered module* $\mathcal{M}_{u,\sigma,q,1}$.

This verifies the analogue of conjecture 1.3.2 in [A] in this case.

References

[A] J. Arthur, *On some problems suggested by the trace formula*, LNM **1041** (1983), 1–49.

[B] A. Borel, *Admissible representations of a semisimple group over a local field with fixed vectors under an Iwahori subgroup*, Invent. Math. **35** (1976), 233–259.

[BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Annals of Math. Studies 94 , Princeton University Press, 1980.

[C] W. Casselman, *A new non–unitarity argument for p–adic representations*, Jour. Fac. of Sci., Univ. of Tokyo **28** (1981), 907–928.

[G] V. Ginsburg, *Deligne–Langlands conjecture and representations of Affine Hecke algebras*, preprint.

[HC] Harish–Chandra, *Harmonic analysis on p–adic groups*, Proc. of Symposia in Pure Math. **26** (1973), 167–192.

[KL1] D. Kazhdan and G. Lusztig, *Equivariant K–theory and representations of Hecke algebras II*, Invent. Math. **80** (1985), 209–231.

[KL2] D. Kazhdan and G. Lusztig, *Proof of the Deligne–Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), 153–215.

[Ln] R. Langlands, *Problems in the theory of automorphic forms*, LNM **170** (1970), 18–86.

[Ls1] G. Lusztig, *On a theorem of Benson and Curtis*, Jour. of Alg. **71** (1981), 490–498.

[Ls2] G. Lusztig, *Some examples of square integrable representations of semisimple p–adic groups*, TAMS **277** (1983), 623–653.

[Ls3] G. Lusztig, *Cells in affine Weyl groups IV*, preprint.

[[Ls4] G. Lusztig, *Cuspidal local systems and graded Hecke algebras*, in preparation.

[R] J. Rogawski, *On modules over the Hecke algebra of a p–adic group*, Invent. Math. **79** (1985), 443–465.

[S] A. Silberger, *The Langlands quotient theorem for p–adic groups*, Math. Ann. **236** (1978), 95–104.

[T] J. Tate, *Number theoretic background*, Proc. of Symposia in Pure Math. **33 Part 2** (1979), 3–26.

[V1] D. Vogan, *Representations of Real Reductive Lie Groups*, Birkhauser, Boston–Basel–Stuggart, 1981.

[V2] D. Vogan, *Unitarizablity of certain series of representations*, Annals of Math. **120** (1984), 141–187.