Constructing Modules of Algebra Groups

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Abstract

We describe a recursive method for decomposing the regular representation of an algebra group. The characters we generate include supercharacters as a special case and retain many of their useful properties, such as having disjoint sets of irreducible constituents. As an application, we show that our constructions produce all the irreducible modules of $\mathbf{U}_n(\mathbb{F}_2)$ for $n \leq 12$, and then use the theory to explicitly construct the two irreducible modules of $\mathbf{U}_{13}(\mathbb{F}_2)$ with non-real characters.

1 Introduction

Given a finite group $\mathbf{G}$, the first goal of the representation theorist is to decompose the group algebra $\mathbb{C}\mathbf{G}$ into irreducible submodules, and then to use this decomposition to construct the irreducible characters and conjugacy classes of the group. While for many families of finite groups we can solve this problem exactly, for others it becomes intractable. The groups $\mathbf{U}_n(\mathbb{F}_q)$ of $n \times n$ upper triangular matrices over a finite field with ones on the diagonal serve as one prominent example. Describing the conjugacy classes of these groups for general n and q is a well known "wild" problem, in the sense that any solution would lead to a general description of wild quivers which presumably does not exist. If there is a natural bijection from the conjugacy classes of the group to its irreducible characters, then the character theory of $\mathbf{U}_n(\mathbb{F}_q)$ may be equally indescribable, but we have no reason to believe a priori that such a map exists. Nevertheless, work to date on the characters of $\mathbf{U}_n(\mathbb{F}_q)$ still places us far from the irreducibles.

For such groups we must reformulate our stated goals. Rather than seeking to classify the irreducible characters and conjugacy classes of $\mathbf{G}$, we might instead try to determine a "compatible" pair of partitions $\mathcal{X}, \mathcal{K}$ of the set of irreducible characters $\mathrm{Irr}(\mathbf{G})$ and the set of conjugacy classes of $\mathbf{G}$. By "compatible," we mean that both partitions consist of the same number of disjoint sets, and that the characters $\sigma_X = \sum_{\psi \in X} \psi(1)\psi$ for $X \in \mathcal{X}$ are constant on each union of the sets $K \in \mathcal{K}$. Certainly the ordinary character theory of a group gives rise to such a pair, but we may in general seek coarser partitions. [7] introduced this notion to generalize a particular construction described first by Carlos André and refined by Ning Yan for $\mathbf{U}_n(\mathbb{F}_q)$ [1, 14]. This construction gives a set of characters whose constituent sets partition $\mathrm{Irr}(\mathbf{U}_n(\mathbb{F}_q))$ and which are constant on unions of conjugacy classes indexed by set partitions of n. [7] generalized this particular construction to a larger family of groups known as *algebra groups* and called the resulting character *supercharacters*.

An algebra group $\mathbf{G}$ is a group of the form $\mathbf{G} = 1 + \mathfrak{n}$ for some finite dimensional nilpotent $\mathbb{F}_q$-algebra $\mathfrak{n}$, and the supercharacters of an algebra group $\mathbf{G}$ are a set of characters decomposing the regular representation which approximate $\mathrm{Irr}(\mathbf{G})$. While not in general irreducible, supercharacters display several useful properties which make them a valid substitute for the complete character theory of $\mathbf{G}$. For example:

(i) Each supercharacter uniquely corresponds to a two-sided orbit in some $\mathbb{F}_q$-vector space under a certain left/right action of $\mathbf{G}$.

(ii) Given a representative λ of the two-sided orbit corresponding a particular supercharacter χ, we can determine whether χ is irredicble by inspecting the intersection of left and right one-sided orbits containing λ.

(iii) The set of supercharacters decompose the character of the regular representation of $\mathbf{G}$.

(iv) Distinct supercharacters have no common irreducible constituents.

(v) Each supercharacter is induced from a linear character of an algebra subgroup of $\mathbf{G}$.

In addition, we have an explicit formula for evaluating supercharacters on elements of $\mathbf{G}$, and supercharacters are constant on certain easily described unions of conjugacy classes called superclasses.

In this work, we show how Diaconis and Isaacs' supercharacter theory fits into a more general recursive method for constructing characters of algebra groups. This method allows us to decompose reducible supercharacters, and provides a way of constructing a set of characters which better approximates $\mathrm{Irr}(\mathbf{G})$ than supercharacters. More importantly, the characters we shall construct will retain analogues of properties (i)-(v). The main tool in these constructions will be a certain kind of recursively defined sequence S indexing a succession of vector spaces and their orbits under a two-sided action of $\mathbf{G}$. If χ_S denotes the character indexed by the sequence S, then our ultimate goal will be to show how to construct a finite rooted tree of sequences $\mathscr{T}$ with the following properties:

(1) The character tree $\hat{\mathscr{T}} = \{\chi_S \mid S \in \mathscr{T}\}$ is uniquely determined by $\mathbf{G}$.

(2) The root of $\mathscr{T}$ indexes the character $\chi_{\mathbf{G}}$ of the regular representation of $\mathbf{G}$, and the sequences in $\mathscr{T}$ which are children of the root index the supercharacters of $\mathbf{G}$.

(3) For each $S \in \mathscr{T}$, the character χ_S is induced from a linear character of an algebra subgroup of $\mathbf{G}$.

(4) If $S \in \mathscr{T}$ and $\mathscr{L}_S \subseteq \mathscr{T}$ is the set of leaf nodes which are descendants of S, then the character χ_S decomposes as the sum

$$\chi_S = \frac{1}{m_S} \sum_{T \in \mathscr{L}_S} m_T \chi_T$$

for some positive integers m_S, m_T. In particular, the character $\chi_{\mathbf{G}}$ of the regular representation of $\mathbf{G}$ decomposes as the sum

$$\chi_{\mathbf{G}} = \sum_{S \in \mathscr{L}} m_S \chi_S$$

where $\mathscr{L}$ denotes the set of leaf nodes in $\mathscr{T}$.

(5) If $S, T \in \mathscr{T}$, then $\langle \chi_S, \chi_T \rangle = 0$ unless T is a descendent of S or vice versa. Thus every irreducible character of $\mathbf{G}$ appears as a constituent of χ_S for exactly one $S \in \mathscr{L}$.

After developing this theory, we discuss how it can be used in practice to compute modules and characters of an algebra group. In particular, we show that our constructions generate all irreducible characters of the upper triangular matrix groups $\mathbf{U}_n(\mathbb{F}_2)$ for $n \leq 12$, and as a final application, we give an explicit construction for the two irreducible characters of $\mathbf{U}_{13}(\mathbb{F}_2)$ with non-real values. The somewhat unexpected existence of these characters was first shown non-constructively in [10, 11]. We are able to prove that these characters are induced from linear characters of the algebra group

$$\mathbf{H} = \left\{ \left(\begin{array}{ccccccccccccc} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 1 & a & \bullet & a & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 1 & b & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & 1 & 0 & 0 & 0 & c & \bullet & 0 & \bullet & \bullet & \bullet \\ & & & & 1 & b & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & 1 & b & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & & 1 & c & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & & & 1 & \bullet & 0 & 0 & \bullet & \bullet \\ & & & & & & & & 1 & 0 & 0 & 0 & \bullet \\ & & & & & & & & & 1 & \bullet & \bullet & \bullet \\ & & & & & & & & & & 1 & \bullet & \bullet \\ & & & & & & & & & & & 1 & \bullet \\ & & & & & & & & & & & & 1 \end{array} \right) \mid a, b, c, \bullet \in \mathbb{F}_2 \right\},$$

where we use the symbol $\bullet$ to label a position whose value can be chosen independently of all other positions. In the process of this construction we answer a number of questions regarding the various properties of these characters. In particular, our observations show that the two non-real irreducible characters of $\mathbf{U}_{13}(\mathbb{F}_2)$ have degree 2^{16}, as conjectured in [11], and hence that all real-valued characters of $\mathbf{U}_{13}(\mathbb{F}_2)$ are realizable over $\mathbb{R}$.

2 Notation

Below, we list our main notations.

Notation	Usage
$\mathbb{F}_q,\ \mathbb{F}_q^+,\ \mathbb{F}_q^\times$	The finite field of $q = p^a$ elements, $\mathbb{F}_q$ viewed as an additive group, and the multiplicative group of nonzero elements.
$\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$	A fixed nontrivial homomorphism $\mathbb{F}_q^+ \to \mathbb{C}^\times$.
$\mathfrak{n}$	A nilpotent finite dimensional $\mathbb{F}_q$-algebra.
$\mathbf{G} = 1 + \mathfrak{n}$	The algebra group of unipotent matrices $1 + X$ for $X \in \mathfrak{n}$.
$\mathfrak{u}_n(\mathbb{F}_q)$	The nilpotent $\mathbb{F}_q$-algebra of $n \times n$ strictly upper triangular matrices.
$\mathbf{U}_n(\mathbb{F}_q) = 1 + \mathfrak{u}_n(\mathbb{F}_q)$	The algebra group of $n \times n$ unipotent upper triangular matrices.
$\mathrm{Irr}(\mathbf{G})$	The set of irreducible characters of a group $\mathbf{G}$.
$\mathfrak{h}_i$	A finite dimensional vector space over $\mathbb{F}_q$ indexed by $i \in \mathbb{N}$.
λ_i	An element of $\mathfrak{h}_i$.
$\mathbf{L}_i,\ \mathbf{R}_i$	Subgroups of $\mathbf{G}$ which act on the $\mathfrak{h}_i$ on the left and right.
$*_i, \circledast_i : \mathbf{L}_i \times \mathfrak{h}_i \to \mathfrak{h}_i$	Two left actions of $\mathbf{L}_i$ on $\mathfrak{h}_i$, usually abbreviated by $*, \circledast$.
$*_i, \circledast_i : \mathfrak{h}_i \times \mathbf{R}_i \to \mathfrak{h}_i$	Two right actions of $\mathbf{R}_i$ on $\mathfrak{h}_i$, usually abbreviated by $*, \circledast$.
$\mathfrak{i}_i$	A subspace of $\mathfrak{h}_i$ determined by λ_i and the $\circledast$-actions of $\mathbf{L}_i$ and $\mathbf{R}_i$.
$S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$	A finite sequence of 5-tuples $(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)$ defined by a specific recursive construction depending on $\mathbf{G}$. We call such an object a *decomposition sequence* of $\mathbf{G}$.
$\mathbf{LStab}_S(\lambda)$	Given $\lambda \in \mathfrak{h}_i$, the subgroup of elements $g \in \mathbf{L}_i$ with $g \circledast_i \lambda_i = \lambda_i$.
$\mathbf{RStab}_S(\lambda)$	Given $\lambda \in \mathfrak{h}_i$, the subgroup of elements $h \in \mathbf{R}_i$ with $\lambda_i \circledast_i h_i = \lambda_i$.
$\mathfrak{h}_S$	The dual space $\mathfrak{i}_n^*$ of $\mathbb{F}_q$-linear functionals of $\mathfrak{i}_n$.
$\mathbf{L}_S,\ \mathbf{R}_S$	The subgroups of $\mathbf{L}_n$, $\mathbf{R}_n$ which would appear in the final term of any decomposition sequence formed by adding one 5-tuple to S.
$v_{S,i} : \mathfrak{h}_i \to \mathbb{C}\mathbf{G}$	A map indexing vectors in $\mathbb{C}\mathbf{G}$, usually abbreviated by v_S.
V_S, M_S	The left and two-sided ideals of $\mathbb{C}\mathbf{G}$ generated by $v_{S,n}(\lambda_n)$.
χ_S	The character of the $\mathbb{C}\mathbf{G}$-module V_S.
$\alpha_{S,i} : \mathbf{L}_i \times \mathfrak{h}_i \to \mathbb{C}^\times$	A coefficient map corresponding to left multiplication by $\mathbf{L}_i$ in V_S.
$\beta_{S,i} : \mathfrak{h}_i \times \mathbf{R}_i \to \mathbb{C}^\times$	A coefficient map corresponding to right multiplication by $\mathbf{R}_i$ in V_S.
$\mathscr{D}(\mathbf{G})$	The set of decomposition sequences of an algebra groups $\mathbf{G}$.
$\mathscr{T}$	A subtree of $\mathscr{D}(\mathbf{G})$ defined by a specific recursive construction.
$\hat{\mathscr{T}}$	The tree of characters formed by replacing each $S \in \mathscr{T}$ with χ_S.

3 Algebra groups

In this and the following section, we review the definition of an algebra group and the construction of its supercharacters. To provide a concrete example, we then briefly describe the supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$.

Fix a finite field $\mathbb{F}_q$ of order q and prime characteristic p. For our purposes, an $\mathbb{F}_q$*-algebra* $\mathfrak{a}$ is a vector space over $\mathbb{F}_q$ with an associative multiplication $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$. For example, matrix multiplication makes the vector space of $n \times n$ matrices over $\mathbb{F}_q$ into an algebra. We say that $\mathfrak{a}$ is *nilpotent* if for every $X \in \mathfrak{a}$ we have $X^k = 0$ for some positive integer k. Throughout this work, we let $\mathfrak{n}$ denote a finite dimensional nilpotent $\mathbb{F}_q$-algebra and let $\mathfrak{n}^*$ denote its dual space of $\mathbb{F}_q$-linear functionals $\lambda : \mathfrak{n} \to \mathbb{F}_q$. The vector space $\mathfrak{u}_n(\mathbb{F}_q)$ of $n \times n$ upper triangular matrices over $\mathbb{F}_q$ with zeros on the diagonal forms a canonical example of such an algebra. Indeed, by Engel's theorem we may always view $\mathfrak{n}$ as a subalgebra of $\mathfrak{u}_n(\mathbb{F}_q)$ for some choice of n and q.

The *algebra group* $\mathbf{G}$ corresponding to $\mathfrak{n}$ is the set of formal sums $\mathbf{G} = 1 + \mathfrak{n} = \{1 + X \mid X \in \mathfrak{n}\}$ with multiplication defined by

$$(1+X)(1+Y) = 1 + (X + Y + XY), \qquad \text{for } X, Y \in \mathfrak{n}.$$

$\mathbf{G}$ is a p-group of size $|\mathbf{G}| = |\mathfrak{n}| = q^d$ where $d = \dim(\mathfrak{n})$ is the dimension of $\mathfrak{n}$ as a vector space, and as a result every algebra group is nilpotent. Despite this, the commutator subgroup of $\mathbf{G}$ may fail to be itself an algebra group [12], and consequently not every p-group is an algebra group. Since we can embed every finite dimensional nilpotent $\mathbb{F}_q$-algebra in $\mathfrak{u}_n(\mathbb{F}_q)$ for some n, we can likewise view every algebra group over $\mathbb{F}_q$ as a subgroup of $\mathbf{U}_n(\mathbb{F}_q) = 1 + \mathfrak{u}_n(\mathbb{F}_q)$, the group of $n \times n$ upper triangular matrices over $\mathbb{F}_q$ with ones on the diagonal. Given this identification, we can always define an algebra group by drawing a matrix with ones on the diagonal, and then writing a linear function in some number of variables in each position above the diagonal to specify its possible values. Of course, we cannot pick arbitrary functions since our underlying nilpotent algebra must be closed, but this working definition provides a useful way of visualizing such groups. For example, we could write

$$\mathbf{G} = \left\{ \begin{pmatrix} 1 & t_1 & t_2 & t_3 & t_4 \\ & 1 & t_1 + t_8 & t_6 & t_7 \\ & & 1 & 0 & t_1 - t_8 \\ & & & 1 & t_8 \\ & & & & 1 \end{pmatrix} \mid t_i \in \mathbb{F}_q \right\} = \left\{ \begin{pmatrix} 1 & a & \bullet & \bullet & \bullet \\ & 1 & a+b & \bullet & \bullet \\ & & 1 & 0 & a-b \\ & & & 1 & b \\ & & & & 1 \end{pmatrix} \mid a, b, \bullet \in \mathbb{F}_q \right\}$$

to define an algebra group over $\mathbb{F}_q$. On the right hand side we use the symbol $\bullet$ to label a position whose value in an element of $\mathbf{G}$ can be chosen independently of all other positions. We use this convention throughout in group definitions to simplify notation.

For a number of reasons, classifying the conjugacy classes and irreducible representations of algebra groups appears to be a very difficult or even impossible task. This should not be terribly surprising; after all, the family in question forms a large class of p-groups and the problem of understanding the representations of all p-groups nearly amounts to understanding the representations of all groups. More strikingly, the conjugacy classes and irreducible representations of many more tangible subfamilies of algebra groups still appear to to defy any general description. The groups $\mathbf{U}_n(\mathbb{F}_q)$ present perhaps the most famous example of this complexity. Results of quiver theory imply that explicitly computing the conjugacy classes and irreducible representations of $\mathbf{U}_n(\mathbb{F}_q)$ amounts to a "wild" problem, although the difference between "wild" and "provably impossible" is a subtle distinction beyond the scope of this work. Likewise, we do not have tight bounds on the asymptotic growth of the number of conjugacy classes, and we appear to be still far from a proof or disproof of Thompson's conjecture that the number of conjugacy classes for fixed n is a polynomial in q.

Despite such obstacles, a series of mathematicians have made significant progress in understanding the representation theory of $\mathbf{U}_n(\mathbb{F}_q)$ and algebra groups in general during last two decades. In particular, Carlos André [1] followed by Ning Yan [14] indepedently developed an elegant approximate character theory for $\mathbf{U}_n(\mathbb{F}_q)$. This theory describes the construction of a set of characters of $\mathbf{U}_n(\mathbb{F}_q)$ whose constituents partition the irreducible characters the group, and which possess a natural combinatorial interpretation in terms of set partitions. At the same time, the theory also describes an equally elegant partition of the conjugacy classes of $\mathbf{U}_n(\mathbb{F}_q)$ on which these characters are constant and easily evaluated. Persi Diaconis and I. Martin Isaacs [7] generalized these constructions from the groups $\mathbf{U}_n(\mathbb{F}_q)$ to all algebra groups. The characters in this theory are called *supercharacters* while the unions of conjugacy classes are called *superclasses*. The body of results surrounding their study has to come to form "supercharacter theory."

In this work, we show how supercharacters fit into a larger recursive construction for partitioning the irreducible characters of an algebra group into disjoint constituents of the regular representation. In order to motivate our theory and to introduce preliminary definitions and theorems, we briefly describe the relevant results from [7].

4 Supercharacters

At its heart supercharacter theory is an attempt to extend Kirillov's orbit method for Lie algebras to finite groups. In this spirit, we construct the supercharacters of an algebra group $\mathbf{G} = 1 + \mathfrak{n}$ by first defining appropriate actions of $\mathbf{G}$ on $\mathfrak{n}$ and $\mathfrak{n}^*$, and then translating information about the orbits of these actions into representation theoretic terms.

To this end, first fix an algebra group $\mathbf{G} = 1 + \mathfrak{n}$. Since we can view $\mathbf{G}$ as a subgroup of the group of units of the algebra $\mathbb{F}_q \cdot 1 + \mathfrak{n}$, left and right multiplication defines a natural left and right action of $\mathbf{G}$ on $\mathfrak{n}$. Similarly, $\mathbf{G}$ acts on $\mathfrak{n}^*$ on the left and right by

$$g\lambda(X) = \lambda(g^{-1}X),$$
$$\lambda h(X) = \lambda(Xh^{-1}),$$

for $g \in \mathbf{G}$, $\lambda \in \mathfrak{n}^*$, $X \in \mathfrak{n}$. More precisely, if we define a linear functional $g\lambda$ by the preceding formula for $g \in \mathbf{G}$ and $\lambda \in \mathfrak{n}^*$, then the map $(g, \lambda) \mapsto g\lambda$ defines a left group action. Both of these left/right actions are *compatible* in the sense that $(gX)h = g(Xh)$ and $(g\lambda)h = g(\lambda h)$ for all $g, h \in \mathbf{G}$, $X \in \mathfrak{n}$, and $\lambda \in \mathfrak{n}^*$. This property makes the notation $gXh = (gX)h = g(Xh)$ and $g\lambda h = (g\lambda)h = g(\lambda h)$ well-defined. It also allows us to describe the sizes of the two-sided orbits corresponding to our left/right actions in terms of the one-sided orbits. The following lemma clarifies what we mean:

Lemma 4.1. Let $\mathbf{L}$ and $\mathbf{R}$ be finite groups which act on a set $\mathcal{X}$ on the left and right, respectively. Suppose the actions of $\mathbf{L}$ and $\mathbf{R}$ are compatible, so that $(gx)h = g(xh)$ for all $g \in \mathbf{L}$, $h \in \mathbf{R}$, and $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ be an arbitrary element and write $\mathbf{L}x\mathbf{R} = \{gxh \mid g \in \mathbf{L}, h \in \mathbf{R}\}$ to denote its two-sided orbit. Then

$$|\mathbf{L}x\mathbf{R}| = \frac{|\mathbf{L}x||x\mathbf{R}|}{|\mathbf{L}x \cap x\mathbf{R}|}.$$

Proof. The two-sided orbit $\mathbf{L}x\mathbf{R}$ is a union of left orbits which are transitively permuted by the right action of $\mathbf{R}$. Because these orbits are transitively permuted, they are all the same size $|\mathbf{L}x|$ and so $|\mathbf{L}x\mathbf{R}| = k|\mathbf{L}x|$ where k is the number of left orbits in $\mathbf{L}x\mathbf{R}$. The right action of $\mathbf{R}$ is transitive on the set of intersections of $x\mathbf{R}$ with the k right translates of $\mathbf{R}x$, and these k sets of cardinality $|\mathbf{L}x \cap x\mathbf{R}|$ partition $|x\mathbf{R}|$. It follows that $|x\mathbf{R}| = k|\mathbf{L}x \cap x\mathbf{R}|$, and so $k = |x\mathbf{R}|/|\mathbf{L}x \cap x\mathbf{R}|$ and $|\mathbf{L}x\mathbf{R}| = |\mathbf{L}x||x\mathbf{R}|/|\mathbf{L}x \cap x\mathbf{R}|$. $\square$

We have a natural bijection $\mathfrak{n} \to \mathbf{G}$ given by $X \mapsto 1 + X \in \mathbf{G}$ for $X \in \mathfrak{n}$. We define the *superclasses* of $\mathbf{G}$ as the sets formed by applying this map to the two-sided $\mathbf{G}$-orbits in $\mathfrak{n}$. More exactly, the superclass of $\mathbf{G}$ containing $g \in \mathbf{G}$ is the set $\mathcal{K}_g = \{1+x(g-1)y \mid x, y \in \mathbf{G}\}$. Since $xg^{-1}x^{-1} = 1+x(g^{-1}-1)x^{-1}$ for all $g, x \in \mathbf{G}$, a superclass is clearly a union of conjugacy classes, and since $1+x(1-1)x^{-1}=1$, there exists a superclass consisting of just the identity element of $\mathbf{G}$.

Constructing the supercharacters of $\mathbf{G}$ requires slightly more exertion on our part. For this, we first choose an arbitrary nontrivial homomorphism $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$; in other words, we let θ denote a non-principal linear characters of the additive group of the field $\mathbb{F}_q$. There is really no natural choice of θ. We might constructively define θ by choosing a basis $\mathcal{B}$ for $\mathbb{F}_q$ as a vector space (i.e., a minimal additive generating set) and then setting $\theta(b) = e^{2\pi i/p}$ for each $b \in \mathcal{B}$. Of course, this still requires a more or less arbitrary choice of basis, although if q is prime then a "natural" choice is $\mathcal{B} = \{1\}$.

Given θ, we define the vector $v_\mu \in \mathbb{C}\mathbf{G}$ for $\mu \in \mathfrak{n}^*$ by

$$v_\mu = \frac{1}{|\mathfrak{n}|} \sum_{X\in\mathfrak{n}} \theta(\mu(X)) \underbrace{(1+X)}_{\in\mathbf{G}}. \tag{4.1}$$

$\mathbf{G}$ acts on $\mathbb{C}\mathbf{G}$ on the left and right by component-wise multiplication, and under these natural actions it is easy to see that

$$gv_\mu = \theta_\mu(g) v_{g\mu} \qquad \text{and} \qquad v_\mu h = \theta_\mu(h) v_{\mu h}, \qquad \text{for } g,h \in \mathbf{G}$$

where $\theta_\mu(g) = \theta\left(\mu(g^{-1}-1)\right)$. Thus, given $\lambda \in \mathfrak{n}^*$, we can naturally define a left module V^λ and a two-sided ideal M^λ by

$$\begin{aligned} V^\lambda &= \mathbb{C}\mathbf{G}v_\lambda = \mathbb{C}\text{-span}\{v_\mu \mid \mu \in \mathbf{G}\lambda\}, \\ M^\lambda &= \mathbb{C}\mathbf{G}v_\lambda\mathbb{C}\mathbf{G} = \mathbb{C}\text{-span}\{v_\mu \mid \mu \in \mathbf{G}\lambda\mathbf{G}\}. \end{aligned} \tag{4.2}$$

Let χ^λ denote the character of the module V^λ. If $\mathbf{LStab}(\lambda) = \{g \in \mathbf{G} \mid g\lambda = \lambda\}$, then $\theta_\lambda : \mathbf{G} \to \mathbb{C}^\times$ restricts to a linear character $\tau : \mathbf{LStab}(\lambda) \to \mathbb{C}^\times$ of $\mathbf{LStab}(\lambda)$ and it follows immediately from our definition that $\chi^\lambda = \tau^{\mathbf{G}}$ is the character induced from τ. As such $\chi(1) = \dim(V^\lambda) = \frac{|\mathbf{G}|}{|\mathbf{LStab}(\lambda)|} = |\mathbf{G}\lambda|$.

We define the *supercharacters* $\mathbf{G}$ as the characters χ^λ for $\lambda \in \mathfrak{n}^*$. Clearly many of these characters are the same. In particular, the map $v_\mu \mapsto v_\mu x$ for $x \in \mathbf{G}$ gives an isomorphism $V^\lambda \to V^{\lambda x}$, and so if $\gamma \in \mathbf{G}\lambda\mathbf{G}$, then $V^\gamma \cong V^\lambda$ and $\chi^\gamma = \chi^\lambda$. It follows from this observation that M^λ affords the character $m_\lambda\chi^\lambda$ where $m_\lambda = \frac{|\mathbf{G}\lambda\mathbf{G}|}{|\mathbf{G}\lambda|} = \frac{|\lambda\mathbf{G}|}{|\mathbf{G}\lambda\cap\lambda\mathbf{G}|}$ is the number of distinct left orbits in the two-sided orbit $\mathbf{G}\lambda\mathbf{G}$. Hence the supercharacters of G are indexed by the distinct two-sided orbits in $\mathfrak{n}^*$. In particular, if $\mathscr{I} \subseteq \mathfrak{n}^*$ is a set of representatives of these orbits and $e_\lambda = \sum_{\gamma\in\mathbf{G}\lambda\mathbf{G}} v_\gamma$ for $\lambda \in \mathfrak{n}^*$, then $\sum_{\lambda\in\mathscr{I}} e_\lambda = 1 \in \mathbb{C}\mathbf{G}$. Since M^λ is a two-sided ideal in $\mathbb{C}\mathbf{G}$, it follows that the irreducible constituents of the modules M^λ for $\lambda \in \mathscr{I}$ partition $\mathrm{Irr}(\mathbf{G})$, and so by an elementary result in character theory we can write

$$e_\lambda = \frac{1}{|\mathbf{G}|} \sum_{\gamma\in\mathbf{G}\lambda\mathbf{G}} \sum_{g\in\mathbf{G}} \theta(\gamma(g-1))g = \sum_{g\in\mathbf{G}} m_\lambda \chi^\lambda(g^{-1})g.$$

By comparing coefficients, this gives the following formula for χ^λ:

$$\chi^\lambda(g) = \frac{1}{m_\lambda} \sum_{\mu\in\mathbf{G}\lambda\mathbf{G}} \theta_\mu(g), \qquad \text{for } g \in \mathbf{G}, \tag{4.3}$$

where, we recall, $m_\lambda = \frac{|\mathbf{G}\lambda\mathbf{G}|}{|\mathbf{G}\lambda|}$ and $\theta_\mu(g) = \theta(\mu(g^{-1}-1))$. In addition, it follows that the group algebra $\mathbb{C}\mathbf{G}$ decomposes as

$$\mathbb{C}\mathbf{G} = \bigoplus_{\lambda\in\mathscr{I}} M^\lambda = \bigoplus_{\lambda\in\mathscr{I}} (V^\lambda)^{\oplus m_\lambda} \tag{4.4}$$

and the character $\chi_{\mathbf{G}}$ of $\mathbb{C}\mathbf{G}$ decomposes as

$$\chi_{\mathbf{G}} = \sum_{\lambda\in\mathscr{I}} m_\lambda\chi^\lambda. \tag{4.5}$$

Finally, by dimensional considerations we have

$$\langle \chi^\lambda, \chi^\mu \rangle = \begin{cases} |\mathbf{G}\lambda \cap \lambda\mathbf{G}|, & \text{if } \mu \in \mathbf{G}\lambda\mathbf{G}, \\ 0, & \text{otherwise,} \end{cases} \tag{4.6}$$

for $\lambda, \mu \in \mathfrak{n}^*$, where $\langle\ ,\ \rangle$ denotes the familiar inner product on the class functions of $\mathbf{G}$ given by

$$\langle \chi, \psi \rangle = \frac{1}{|\mathbf{G}|} \sum_{g \in \mathbf{G}} \chi(g)\overline{\psi(g)}.$$

Thus distinct supercharacters have disjoint irreducible constituents, and we have a simple condition for determining whether or not a supercharacter is itself irreducible.

5 Supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$

In this section we describe the superclasses and supercharacters of the group $\mathbf{U}_n(\mathbb{F}_q) = 1 + \mathfrak{u}_n(\mathbb{F}_q)$ of $n \times n$ upper triangular matrices over $\mathbb{F}_q$ with ones on the diagonal. These characters and classes posess a particularly elegant combinatorial interpretation, and we will often refer to them later in examples.

The superclasses and supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$ are indexed by $\mathbb{F}_q$-labeled set partitions. A *set partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ of n is a finite sequence of disjoint, nonempty subsets $\lambda_i \subseteq \{1, \dots, n\}$ such that

$$\bigcup \lambda_i = \{1, 2, \dots, n\} \qquad \text{and} \qquad \min(\lambda_1) < \min(\lambda_2) < \dots$$

The sets λ_i are called the *parts* of λ. We view each part as a finite increasing sequence of positive integers, and typically abbreviate λ by writing the numbers in each part from left to write, separating successive parts with the "|" symbol. For example, we write $\lambda = (\{1,2\},\{3\})$ as $\lambda = 12|3$. Given $i, j \in \{1, \dots, n\}$ and a set partition λ of n, we say that the position (i,j) is in the *support* of λ if i, j are contained in the same part λ_k with $i < j$, such that there is no $x \in \lambda_k$ with $i < x < j$. We denote the set of such (i,j) by $\mathrm{supp}(\lambda)$. For example, $\mathrm{supp}(145|2|36) = \{(1,4),(4,5),(3,6)\}$.

An $\mathbb{F}_q$*-labeled set partition* is a set partition λ whose support is labeled by elements of $\mathbb{F}_q^\times$. We write a labeled set partition by writing the set partition λ as above, and then replacing each supported point "ij" with "$i \overset{t}{\frown} j$" where $t \in \mathbb{F}_q^\times$ is the label assigned to (i,j). Since $\mathbb{F}_q^\times$ has only one element, set partitions over $\mathbb{F}_2$ are equivalent to unlabeled set partitions. An example of an $\mathbb{F}_3$-labeled set partition of 5 is

$$\lambda = 1 \overset{1}{\frown} 3 \overset{2}{\frown} 5 | 2 \overset{1}{\frown} 4. \tag{5.1}$$

Let $\mathscr{P}_n(\mathbb{F}_q)$ denote the set of $\mathbb{F}_q$-labeled set partitions of n. We denote the number of elements of $\mathscr{P}_n(\mathbb{F}_q)$ by $B(n,q)$; this quantity is given by the recursive formula

$$\begin{aligned} &B(0,q) = 1, \\ &B(n+1,q) = \sum_{k=0}^{n} \binom{n}{k} (q-1)^k B(n-k,q), \qquad \text{for } n \geq 0. \end{aligned} \tag{5.2}$$

For $q = 2$, $B(n,q)$ is equal to the nth Bell number.

For each $(i,j) \in \mathrm{supp}(\lambda)$, let $\lambda_{ij} \in \mathbb{F}_q^\times$ denote the corresponding label, and for each $(i,j) \notin \mathrm{supp}(\lambda)$ let $\lambda_{ik} = 0$. With this notation, we can naturally view the set partitions λ of n as $n \times n$ upper triangular matrices over $\mathbb{F}_q$ whose (i,j)th entry is λ_{ij}. We refer to this matrix representation of a set partition as its *diagram*. For example, the diagram of the set partition given in (5.1) is

$$\lambda = 1 \overset{1}{\frown} 3 \overset{2}{\frown} 5 | 2 \overset{1}{\frown} 4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can view $\mathscr{P}_n(\mathbb{F}_q)$ as a subset of either $\mathfrak{u}_n(\mathbb{F}_q)$ or $\mathfrak{u}_n^*(\mathbb{F}_q)$ by identifying set partitions with their diagrams. As shown in [14], under these identifications $\mathscr{P}_n(\mathbb{F}_q)$ comprises a complete set of representatives of the two-sided orbits in both $\mathfrak{u}_n(\mathbb{F}_q)$ and $\mathfrak{u}_n^*(\mathbb{F}_q)$. Therefore $\mathbb{F}_q$-labeled set partitions index both the superclasses and supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$.

Following the convention of [6, 14], we define the *intertwining index* $\iota(\lambda)$ of a set partition $\lambda \in \mathscr{P}_n(\mathbb{F}_q)$ as the number of pairs of positions $(i,k),(j,l) \in \mathrm{supp}(\lambda)$ with $1 \leq i < j < k < l \leq n$. For example, $\iota(135|24) = 2$ since we have $(1,3),(2,4) \in \mathrm{supp}(135|24)$ and $(2,4),(3,5) \in \mathrm{supp}(135|24)$. As shown in [6, 7, 14], it follows directly from inspecting the diagram of $\lambda \in \mathscr{P}_n(\mathbb{F}_q)$ that $\langle \chi^\lambda, \chi^\lambda \rangle = |\mathbf{U}_n(\mathbb{F}_q)\lambda \cap \lambda \mathbf{U}_n(\mathbb{F}_q)| = q^{\iota(\lambda)}$, and hence that χ^λ is irreducible if and only if $\iota(\lambda) = 0$. A set partition satisfying this condition is called *non-crossing*, and so the irreducible supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$ are indexed by non-crossing $\mathbb{F}_q$-labeled set partitions. More intuitively, if we write the numbers $\{1,\ldots,n\}$ in a circle, then a set partition λ can be represented by drawing an $\mathbb{F}_q$-labeled line between each pair of numbers $(i,j) \in \mathrm{supp}(\lambda)$. A non-crossing set partition is then a partition which can be drawn in this way with no crossing segments. Let $\mathscr{N}_n(\mathbb{F}_q)$ denote the set of non-crossing $\mathbb{F}_q$-labeled set partitions. We denote the number of elements of $\mathscr{N}_n(\mathbb{F}_q)$ by $C(n,q)$; this quantity is given by the recursive formula

$$\begin{aligned} &C(0,q) = 1, \\ &C(n,q) = \sum_{k=0}^{n-1} (q-1)^k N(n, n-k), \qquad \text{for } n \geq 1, \end{aligned} \tag{5.3}$$

where $N(n,k)$ denotes the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n \geq 1$ and $k \in \{1,\ldots,n\}$. For $q = 2$, $C(n,q) = \frac{1}{n+1}\binom{2n}{n}$ is equal to the nth Catalan number.

Using these constructions, we can slightly improve the upper bound given in [8] on the number of conjugacy classes of $\mathbf{U}_n(\mathbb{F}_q)$ as a function of n. It appears the the following theorem gives the best known bounds on this quantity:

Theorem 5.1. If the number of conjugacy classes of $\mathbf{U}_n(\mathbb{F}_q)$ is written $q^{A(n,q)n^2}$, where $A(n,q)$ depends on n and q, then

$$\frac{1}{12} - \epsilon_n \leq A(n,q) \leq \frac{2}{12} + \epsilon_n$$

where ϵ_n is a quantity tending to zero as $n \to \infty$.

Remark. Theorem 3.2 of [8] gives a bound of $\frac{1}{12} - \epsilon_n \leq A(n,q) \leq \frac{3}{12} + \epsilon_n$.

Proof. The lower bound is given in [8]. Since the support of any $\lambda \in \mathscr{P}_n(\mathbb{F}_q)$ contains at most $n-1$ positions, there are at most $(q-1)^{n-1}B(n,2)$ distinct supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$. The number of irreducible constituents of any supercharacter χ^λ is at most $\langle \chi^\lambda, \chi^\lambda \rangle = q^{\iota(\lambda)}$, and every irreducible character of the group appears in some supercharacter. As $(q-1)^{n-1}B(n,2) \leq q^n n^n = q^{n \log_q n + n} = q^{\epsilon_n n^2}$ for $q \geq 2$, it follows that the number of irreducible characters, which is the same as the number of conjugacy classes, is at most $q^{M(n) + \epsilon_n n^2}$ where $M(n) = \max_{\lambda \in \mathscr{P}_n(\mathbb{F}_q)} \iota(\lambda)$. Thus to prove the given result we must show that $M(n) \leq \left(\frac{1}{6} + \epsilon_n\right) n^2$.

To this end, choose $\lambda \in \mathscr{P}_n(\mathbb{F}_q)$ such that $M(n) = \iota(\lambda)$. Suppose $(i,k_1),(j,k_2) \in \mathrm{supp}(\lambda)$, where $i < j$, and assume there are no positions in rows $i+1,\ldots,j-1$, In this case we must have $k_1 < k_2$ since otherwise, by switching positions $(i, k_1), (j, k_2)$ in $\mathrm{supp}(\lambda)$ with $(i, k_2), (j, k_1)$, we could obtain a new set partition $\lambda' \in \mathscr{P}_n(\mathbb{F}_q)$ with $\iota(\lambda') = 1 + \iota(\lambda)$. Thus we may assume that $\mathrm{supp}(\lambda)$ is of the form $\mathrm{supp}(\lambda) = \{(i_1, k_1), \dots, (i_m, k_m)\}$ where $1 \leq i_1 < \cdots < i_m < n$ and $1 < k_1 < \cdots < k_m \leq n$. We next observe that if $i_{j+1} - i_j = \delta > 1$ for some $j = 1, \dots, m$, then by replacing positions $(i_{j+1}, k_{j+1}), \dots, (i_m, k_m)$ in $\mathrm{supp}(\lambda)$ with $(i_{j+1} - \delta + 1, k_{j+1}), \dots, (i_m - \delta + 1, k_m)$ we could obtain a new set partition $\lambda' \in \mathscr{P}_n(\mathbb{F}_q)$ for which $\iota(\lambda) \geq \iota(\lambda)$. Thus we may assume without loss of generality that $i_{j+1} = i_j + 1$ for each j. By a symmetric argument, we may likewise assume that $k_{j+1} = k_j + 1$ for each j.

It follows from these observations that the nonzero positions of λ all lie on the same diagonal of an $n \times n$ matrix. Clearly adding positions to this diagonal would only increase $\iota(\lambda)$, and so may assume that $\mathrm{supp}(\lambda)$ is exactly the set of positions on some diagonal of an $n \times n$ matrix. Suppose $\mathrm{supp}(\lambda)$ consists of the positions on the diagonal containing $(1, k)$. We may assume that $k \leq \lfloor n/2 \rfloor + 1$ since otherwise could increase $\iota(\lambda)$ by shifting each nonzero position of λ to the left one column and then adding a position in the last column. In this case it is easily seen by summing the positions on the $k-1$ lower diagonals and then subtracting the positions which are to the left of but not below positions in $\mathrm{supp}(\lambda)$ or vice versa that

$$\iota(\lambda) = \sum_{i=1}^{k-1} (n - i) - k(k-1) = (k-1)(n - \tfrac{3}{2}k).$$

The right hand side achieves a global maximum as a function of k of $\left(\frac{n}{3} - \frac{1}{2}\right)\left(\frac{n}{2} - \frac{3}{4}\right) \leq \left(\frac{1}{6} + \epsilon_n\right) n^2$, so $M(n) \leq \left(\frac{1}{6} + \epsilon_n\right) n^2$ as desired. □

6 Decomposition sequences

In this section we describe how to construct a certain kind of recursively defined sequence which we will use to index a family of modules of an algebra group. The characters of these modules will include supercharacters as a special case, and each step in their construction will very much mirror the supercharacter constructions described in Section 4.

Throughout this section, fix an algebra group $\mathbf{G} = 1 + \mathfrak{n}$. To motivate our ideas, we first present the following definitions. Given $\lambda \in \mathfrak{n}^*$, let $\mathbf{L}_\lambda, \mathbf{R}_\lambda \subseteq \mathbf{G}$ be the sets

$$\mathbf{L}_\lambda = \{g \in \mathbf{G} \mid g\lambda \in \mathbf{G}\lambda \cap \lambda\mathbf{G}\},$$
$$\mathbf{R}_\lambda = \{h \in \mathbf{G} \mid \lambda h \in \mathbf{G}\lambda \cap \lambda\mathbf{G}\}.$$

One can easily check that these sets are in fact subgroups of $\mathbf{G}$, and we will soon show in a slightly more general context that these subgroups are algebra groups. For a moment fix $\lambda \in \mathfrak{n}^*$, and define W as the vector space

$$W = \mathbb{C}\mathbf{L}_\lambda v_\lambda = \mathbb{C}\text{-span}\{v_\mu \mid \mu \in \mathbf{G}\lambda \cap \lambda G\} = v_\lambda \mathbb{C}\mathbf{R}_\lambda.$$

Observe that W is a left $\mathbb{C}\mathbf{L}_\lambda$-module and a right $\mathbb{C}\mathbf{R}_\lambda$-module. The essential motivation for our constructions comes from the following observations:

(1) V^λ is the module $V^\lambda = \mathrm{Ind}_{\mathbf{L}_\lambda}^{\mathbf{G}}(W) = \mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{L}_\lambda} W$ induced from W.

(2) If $U \subseteq W$ is a left $\mathbb{C}\mathbf{L}_\lambda$-submodule and a right $\mathbb{C}\mathbf{R}_\lambda$-submodule of W, then the module $\mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{L}_\lambda} U \otimes_{\mathbb{C}\mathbf{R}_\lambda} \mathbb{C}\mathbf{G}$ is a submodule of M^λ and a two-sided ideal in $\mathbb{C}\mathbf{G}$ which decomposes as a direct sum of isomorphic copies of the induced module $\mathrm{Ind}_{\mathbf{L}_\lambda}^{\mathbf{G}}(U)$.

Both statements follow essentially by inspection, but they contain a powerful idea which we will exploit throughout to decompose the supercharcters of $\mathbf{G}$ into smaller constituents. In particular, in order to construct a finer partition of $\mathrm{Irr}(\mathbf{G})$ than the one afforded by supercharacter theory, we need some kind general method of decomposing a module into left and right invariant submodules. We decompose the group algebra $\mathbb{C}\mathbf{G}$ into supercharacter modules by first noting that $\mathbf{G}$ naturally acts on the vector space $\mathfrak{n}$ on the left and right by matrix multiplication. These actions correspond to a dual action of $\mathbf{G}$ on the vector space $\mathfrak{n}^*$, the orbits of which index two-sided ideals in $\mathbb{C}\mathbf{G}$. To transplant this process to the module W, we observe that the groups $\mathbf{L}_\lambda$ and $\mathbf{R}_\lambda$ naturally act on the vector space $\mathfrak{i} = \mathbf{G}\lambda \cap \lambda\mathbf{G} - \lambda$. We can use these actions to define a dual left and right action of $\mathbf{L}_\lambda$ and $\mathbf{R}_\lambda$ on $\mathfrak{i}^*$, and perhaps the orbits of these actions will similarly index $\mathbf{L}_\lambda$- and $\mathbf{R}_\lambda$-invariant submodules of W. This will indeed turn out to be the case, and so our observations suggest a recursive method for decomposing the supercharacter χ^λ whereby we generate a sequence of smaller and smaller vector spaces whose orbits under certain actions index smaller and smaller submodules of $\mathbb{C}\mathbf{G}$.

This covers the big picture; now we move on to the details. As mentioned earlier, the main aim of this section is to define a kind of recursively constructed sequence S. Each term of this sequence we be a 5-tuple of the form $(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)$. $\mathfrak{h}_i$ is a vector space over $\mathbb{F}_q$ of which $\lambda_i \in \mathfrak{h}_i$ is an arbitrary element; $\mathbf{L}_i, \mathbf{R}_i \subseteq \mathbf{G}$ are subgroups; and $\mathfrak{i}_i \subseteq \mathfrak{h}_i$ is a subspace. The groups $\mathbf{L}_i$ and $\mathbf{R}_i$ act on $\mathfrak{h}_i$ on the left and right in two ways: by a natural multiplication $*_i$, and by a less natural action $\circledast_i$ whose definition depends on previously defined structures. Let $\mathbf{L}_i \circledast_i \lambda$ and $\lambda \circledast_i \mathbf{R}_i$ denote the left and right orbits containing λ under the latter action. We then define $\mathfrak{i}_i$ as the set $\mathfrak{i}_i = \mathbf{L}_i \circledast_i \lambda_i \cap \lambda_i \circledast_i \mathbf{R}_i - \lambda_i$; this set turns out to be a vector space.

To define the next 5-tuple $(\lambda_{i+1}, \mathfrak{h}_{i+1}, \mathbf{L}_{i+1}, \mathbf{R}_{i+1}, \mathfrak{i}_{i+1})$, we let $\mathfrak{h}_{i+1} = \mathfrak{i}_i^*$ be the dual space of $\mathbb{F}_q$-linear functionals on $\mathfrak{i}_i$, and choose an arbitrary element $\lambda_{i+1} \in \mathfrak{h}_{i+1}$. The groups $\mathbf{L}_{i+1} \subseteq \mathbf{L}_i$ and $\mathbf{R}_{i+1} \subseteq \mathbf{L}_i$ are then defined as the subsets of elements which permute that intersection $\mathbf{L}_i \circledast \lambda_i \cap \lambda_i \circledast \mathbf{R}_i$ of the left and right orbit of and λ_i.We have analogous left and right actions $*_{i+1}, \circledast_{i+1}$ of these subgroups on $\mathfrak{h}_{i+1}$, and as before, we define $\mathfrak{i}_{i+1}$ as the vector space $\mathfrak{i}_{i+1} = \mathbf{L}_{i+1} \circledast_{i+1} \lambda_{i+1} \cap \lambda_{i+1} \circledast_{i+1} \mathbf{R}_{i+1} - \lambda_{i+1}$.

Continuing this process, we can define a sequence of 5-tuples $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ of arbitrary length. Even from the cursory outline given above, one can see that S is uniquely determined by the elements $\lambda_i \in \mathfrak{h}_i$. However, to collect all of our notations in one place, we include the auxiliary objects $\mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i$ in each term. We call S a *decomposition sequence* since each successive term specifies how to decompose the $\mathbb{C}\mathbf{G}$ module indexed by the preceding subsequence. The following definition formally presents the recursive construction of a decomposition sequence outlined above, although at this point it should not be clear that these constructions are well-defined.

Definition 1. A *decomposition sequence* $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ of an algebra group $\mathbf{G} = 1 + \mathfrak{n}$ is a sequence of 5-tuples constructed by the following recursive procedure:

(1) Let $\mathfrak{h}_0 = \mathfrak{n}$ and $\mathbf{L}_0 = \mathbf{R}_0 = \mathbf{G}$. Define two left and right actions $*_0, \circledast_0$ of $\mathbf{L}_0$ and $\mathbf{R}_0$ on $\mathfrak{h}_0$ by

$$\begin{aligned} g *_0 \lambda &= g\lambda, \\ \lambda *_0 h &= \lambda h, \end{aligned} \qquad \text{and} \qquad \begin{aligned} g \circledast_0 \lambda &= g(\lambda + 1) - 1, \\ \lambda \circledast_0 h &= (\lambda + 1)h - 1, \end{aligned}$$

for $g \in \mathbf{L}_0$, $h \in \mathbf{R}_0$, $\lambda \in \mathfrak{h}_0$. Let $\lambda_0 = 0 \in \mathfrak{h}_0$ and $\mathfrak{i}_0 = \mathfrak{n}$.

(2) For $0 < i \le n$, let $\mathfrak{h}_i = \mathfrak{i}_{i-1}^*$ be the dual space of $\mathbb{F}_q$-linear functional on $\mathfrak{i}_{i-1}$ and let

$$\begin{aligned} \mathbf{L}_i &= \{g \in \mathbf{L}_{i-1} \mid g \circledast_{i-1} \lambda_{i-1} - \lambda_{i-1} \in \mathfrak{i}_{i-1}\}, \\ \mathbf{R}_i &= \{g \in \mathbf{R}_{i-1} \mid \lambda_{i-1} \circledast_{i-1} g - \lambda_{i-1} \in \mathfrak{i}_{i-1}\}. \end{aligned}$$

Define two left and right actions $*_i, \circledast_i$ of $\mathbf{L}_i$ and $\mathbf{R}_i$ on $\mathfrak{h}_i$ by the formulas

$$\begin{aligned} (g *_i \lambda)(X) &= \lambda(g^{-1} *_{i-1} X), \\ (\lambda *_i h)(X) &= \lambda(X *_{i-1} h^{-1}), \end{aligned} \qquad \text{and} \qquad \begin{aligned} g \circledast_i \lambda &= g *_i \lambda + \Phi^{\mathrm{L}}_{S,i}[g], \\ \lambda \circledast_i h &= \lambda *_i h + \Phi^{\mathrm{R}}_{S,i}[h] \end{aligned}$$

for $g \in \mathbf{L}_i$, $h \in \mathbf{R}_i$, $\lambda \in \mathfrak{h}_i$, and $X \in \mathfrak{i}_{i-1}$, where $\Phi^{\mathrm{L}}_{S,i}[g], \Phi^{\mathrm{R}}_{S,i}[h] \in \mathfrak{h}_i$ are the linear functionals defined by

$$\Phi^{\mathrm{L}}_{S,i}[g](X) = \begin{cases} 0, & i = 1, \\ X(g \circledast_{i-2} \lambda_{i-2} - \lambda_{i-2}), & i > 1, \end{cases}$$

$$\Phi^{\mathrm{R}}_{S,i}[h](X) = \begin{cases} 0, & i = 1, \\ X(\lambda_{i-2} \circledast_{i-2} h - \lambda_{i-2}), & i > 1. \end{cases}$$

Choose an arbitrary element $\lambda_i \in \mathfrak{h}_i$, and let $\mathfrak{i}_i$ be the subspace of $\mathfrak{h}_i$ given by the set $\mathfrak{i}_i = \mathbf{L}_i \circledast_i \lambda_i \cap \lambda_i \circledast_i \mathbf{R}_i - \lambda_i$.

To simplify notation, we typically omit the subscripts from the actions $*_i, \circledast_i$ and write $*, \circledast$ when the context is clear. Since we can think of the vector spaces $\mathfrak{h}_i$ as mutually disjoint, this convention presents minimal ambiguity.

Remark. This definition employs the same symbols $*, \circledast$ to denote our left and right actions for all decomposition sequences S. We will always discuss these actions in the context of some fixed sequence S, so in practice this slightly abusive convention should not cause any confusion.

Let $\mathscr{D}(\mathbf{G})$ denote the set of all decomposition sequences of $\mathbf{G}$. We define the *rank* of a sequence $S \in \mathscr{D}(\mathbf{G})$, denoted $\mathrm{rank}(S)$, to be the least positive integer $r \leq n$ such that $|\mathfrak{i}_r| = |\mathfrak{i}_{r+1}|$, or n if no such r exists. In a precise sense, we shall see that extending a decomposition sequence to a length greater than its rank does not yield any additional information, and so the problem of computing "all" decomposition sequences is a finite one. Before showing that these concepts are well-defined, we provide a detailed example of what a decomposition sequence looks like in practice.

Example. Suppose $\mathbf{G} = 1+\mathfrak{n}$ is the algebra group $\mathbf{U}_7(\mathbb{F}_q) = \mathfrak{u}_7(\mathbb{F}_q)$ of 7×7 upper triangular matrices over $\mathbb{F}_q$:

$$\mathbf{G} = \left\{ \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 1 & \bullet & \bullet & \bullet & \bullet \\ & & & 1 & \bullet & \bullet & \bullet \\ & & & & 1 & \bullet & \bullet \\ & & & & & 1 & \bullet \\ & & & & & & 1 \end{pmatrix} \mid \bullet \in \mathbb{F}_q \right\}.$$

We construct a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^{2}$ with $\mathrm{rank}(S) = 2$. Following Definition 1, we let $\lambda_0 = 0 \in \mathfrak{h}_0 = \mathfrak{i}_0 = \mathfrak{n}$ and $\mathbf{L}_0 = \mathbf{R}_0 = \mathbf{G}$. $\mathfrak{h}_1$ is then given by the dual space $\mathfrak{n}^*$ and $\mathbf{L}_1 = \mathbf{R}_1 = \mathbf{G}$. We can identify $\mathfrak{h}_1 = \mathfrak{n}^*$ with $\mathfrak{i}_0 = \mathfrak{n}$ by viewing each functional $\lambda \in \mathfrak{n}^*$ as a matrix and evaluating λ at $X \in \mathfrak{n}$ by $\lambda(X) = \sum_{i,j} \lambda_{ij} X_{ij}$. Under this identification, let

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}.$$

The left and right orbits of λ_1 under the $\circledast$-actions of $\mathbf{L}_1$ and $\mathbf{R}_1$ are then

$$\mathbf{L}_1 \circledast \lambda_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & \bullet & 1 & 0 & 0 \\ & & 0 & \bullet & \bullet & 1 & 0 \\ & & & 0 & \bullet & \bullet & 1 \\ & & & & 0 & \bullet & \bullet \\ & & & & & 0 & \bullet \\ & & & & & & 0 \end{pmatrix} \right\} \quad \text{and} \quad \lambda_1 \circledast \mathbf{R}_1 = \left\{ \begin{pmatrix} 0 & \bullet & \bullet & 1 & 0 & 0 & 0 \\ & 0 & \bullet & \bullet & 1 & 0 & 0 \\ & & 0 & \bullet & \bullet & 1 & 0 \\ & & & 0 & \bullet & \bullet & 1 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \right\}$$

so

$$\mathfrak{i}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & \bullet & 0 & 0 & 0 \\ & & 0 & \bullet & \bullet & 0 & 0 \\ & & & 0 & \bullet & \bullet & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \mid \bullet \in \mathbb{F}_q \right\}.$$

$\mathfrak{h}_2$ is then the dual space of $\mathfrak{i}_1$, and $\mathbf{L}_2$ and $\mathbf{R}_2$ are given by the algebra subgroups

$$\mathbf{L}_2 = \left\{ \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 1 & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 1 & \bullet & 0 & \bullet & \bullet \\ & & & 1 & 0 & 0 & \bullet \\ & & & & 1 & \bullet & \bullet \\ & & & & & 1 & \bullet \\ & & & & & & 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathbf{R}_2 = \left\{ \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 1 & \bullet & 0 & \bullet & \bullet & \bullet \\ & & 1 & 0 & 0 & \bullet & \bullet \\ & & & 1 & \bullet & \bullet & \bullet \\ & & & & 1 & \bullet & \bullet \\ & & & & & 1 & \bullet \\ & & & & & & 1 \end{pmatrix} \right\}$$

As before we can identity $\mathfrak{h}_2 \cong \mathfrak{i}_1$ by viewing functionals as matrices and evaluating $\lambda(X)$ for $\lambda \in \mathfrak{h}_2$ and $X \in \mathfrak{i}_1$ by means of the standard matrix inner product. Under this identification, let

$$\lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}.$$

The left and right orbits of λ_2 under the $\circledast$-actions of $\mathbf{L}_2$ and $\mathbf{R}_2$ are then

$$\mathbf{L}_2 \circledast \lambda_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & \bullet & 0 & 0 & 0 \\ & & 0 & \bullet & 1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \right\} \quad \text{and} \quad \lambda_2 \circledast \mathbf{R}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & \bullet & \bullet & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} \right\}$$

so $\mathfrak{i}_2 = \{0\}$. One could technically extend the sequence S past this point, but since $\mathfrak{i}_2$ is trivial, all subsequent terms would necessarily be trivial as well. We will show in the next section that this decomposition sequence indexes one of the irreducible constituents of the reducible supercharacter χ^{λ_1} of $\mathbf{U}_7(\mathbb{F}_q)$.

To verify that these constructions are well-defined, we must show that the sets $\mathbf{L}_i, \mathbf{R}_i$ are groups, that the operations $*, \circledast$ define left and right group actions, and that the sets $\mathfrak{i}_i$ are vector spaces. In the course of this verification, we will show in addition that the left/right actions defined by $*_i$ and $\circledast_i$ are compatible. To this end, we fix a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and proceed by induction on i. By definition $\mathbf{L}_0 = \mathbf{R}_0 = \mathbf{G}$ are groups. Since we can view $\mathbf{G}$ as a subgroup of the group of units of the algebra $\mathbb{F}_q \cdot 1 + \mathfrak{n}$, the multiplication $*$ defines compatible left and right actions of $\mathbf{L}_0$ and $\mathbf{R}_0$ on $\mathfrak{h}_0 = \mathfrak{n}$. Likewise, checking that $\circledast$ defines compatible left and right group actions of $\mathbf{L}_0$ and $\mathbf{R}_0$ on $\mathfrak{h}_0$ requires trivial verification, and $\mathfrak{i}_0 = \mathfrak{n}$ is by definition a vector space.

Now let $0 < i \leq n$ and suppose that for all $0 \leq j < i$, the sets $\mathbf{L}_j, \mathbf{R}_j$ are groups, the operations $*, \circledast$ define compatible left and right group actions of $\mathbf{L}_j$ and $\mathbf{R}_j$ on $\mathfrak{h}_j$, and the set $\mathfrak{i}_j$ is a vector space. To show that $\mathbf{L}_i$ is a subgroup of $\mathbf{L}_{i-1}$, suppose $g, h \in \mathbf{L}_i$, so that $g \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{g}$ and $h \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{h}$ for $\tilde{g}, \tilde{h} \in \mathbf{R}_{i-1}$. Since $\circledast$ defines a left and right group action on $\mathfrak{h}_{i-1}$, we have that $h^{-1} \circledast \lambda_{i-1} \circledast \tilde{h} = h^{-1}h \circledast \lambda_{i-1} = \lambda_{i-1}$ so $h^{-1} \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{h}^{-1}$. Therefore $gh^{-1} \in \mathbf{L}_i$ since $gh^{-1} \circledast \lambda_{i-1} = g \circledast \lambda_{i-1} \circledast \tilde{h}^{-1} = \lambda_{i-1} \circledast \tilde{g}\tilde{h}^{-1}$, so $\mathbf{L}_i$

is a subgroup. By the same argument switched from the left to the right action, it follows that $\mathbf{R}_i$ is a subgroup as well.

The operations $*, \circledast$ of $\mathbf{L}_i$ and $\mathbf{R}_i$ on $\mathfrak{h}_i$ are only well-defined if $\mathfrak{i}_{i-1}$ is closed under the $*$-actions of $\mathbf{L}_i$ and $\mathbf{R}_i$; that is, if $g * X \in \mathfrak{i}_{i-1}$ and $X * h \in \mathfrak{i}_{i-1}$ for all $X \in \mathfrak{i}_{i-1}$, $g \in \mathbf{L}_i$, and $h \in \mathbf{R}_i$. We can show that this condition holds by observing that if $0 \leq j < i$ and $\mu, \nu \in \mathfrak{h}_j$, then by definition

$$\begin{aligned} g \circledast (\mu + \nu) &= g * \mu + g \circledast \nu, &\quad \text{for all } g \in \mathbf{L}_j, \\ (\mu + \nu) \circledast h &= \mu * h + \nu \circledast h, &\quad \text{for all } h \in \mathbf{R}_j. \end{aligned} \tag{6.1}$$

Since every $X \in \mathfrak{i}_{i-1}$ is of the form $X = x \circledast \lambda_{i-1} - \lambda_{i-1}$ for some $x \in \mathbf{L}_i$, we have

$$\begin{aligned} g * X &= g * X + g \circledast \lambda_{i-1} - g \circledast \lambda_{i-1} = g \circledast (x \circledast \lambda_{i-1}) + g \circledast \lambda_{i-1} \\ &= (gx \circledast \lambda_{i-1} - \lambda_{i-1}) - (g \circledast \lambda_{i-1} - \lambda_{i-1}) \in \mathfrak{i}_{i-1} \end{aligned}$$

for all $g \in \mathbf{L}_i$, where the inclusion follows from our assumption that $\mathfrak{i}_{i-1}$ is a subspace. A similar argument shows that $X * h \in \mathfrak{i}_{i-1}$ for all $h \in \mathbf{R}_i$, and so $\mathfrak{i}_{i-1}$ is closed under the $*$-actions of $\mathbf{L}_i$ and $\mathbf{R}_i$, as desired.

One can easily show by finite induction that the left and right actions defined by $*$ are linear; i.e., that $g * (aX + bY) * h = a(g * X * h) + b(g * Y * h)$ for all $a, b \in \mathbb{F}_q$, $X, Y \in \mathfrak{i}_{i-1}$, $g \in \mathbf{L}_{i-1}$, and $h \in \mathbf{R}_{i-1}$. As such, it follows immediately that the maps defined by $g * \lambda$, $\lambda * h$ and $g \circledast \lambda$, $\lambda \circledast h$ for $g \in \mathbf{L}_i$, $h \in \mathbf{R}_i$ and $\lambda \in \mathfrak{h}_i$ are indeed well-defined linear functionals on $\mathfrak{i}_{i-1}$. It is a routine exercise to check that $*$ defines compatible left and right group actions of $\mathbf{L}_i$ and $\mathbf{R}_i$ on $\mathfrak{h}_i$, but the same verification for $\circledast$ requires a bit more work.

If $i = 1$ then $\circledast$ and $*$ define the same actions, so assume $i > 1$. Let $g, h \in \mathbf{L}_i$ and $\lambda \in \mathfrak{h}_i$. For all $X \in \mathfrak{i}_{i-1}$ we have

$$\begin{aligned} (g \circledast (h \circledast \lambda))(X) &= (h \circledast \lambda)(g^{-1} * X) + \Phi^{\mathrm{L}}_{S,i}[g](X) \\ &= \lambda(h^{-1}g^{-1} * X) + \Phi^{\mathrm{L}}_{S,i}[g](X) + \Phi^{\mathrm{L}}_{S,i}[h](g^{-1} * X) \\ &= \lambda(h^{-1}g^{-1} * X) + X(g \circledast \lambda_{i-2} - \lambda_{i-2}) + (g^{-1} * X)(h \circledast \lambda_{i-2} - \lambda_{i-2}) \\ &= \lambda(h^{-1}g^{-1} * X) + X(g * (h \circledast \lambda_{i-2} - \lambda_{i-2}) + g \circledast \lambda_{i-2} - \lambda_{i-2}). \end{aligned}$$

To show that this expression is equal to $(gh \circledast \lambda)(X)$, it suffices to prove that

$$g * (h \circledast \lambda_{i-2} - \lambda_{i-2}) + g \circledast \lambda_{i-2} = gh \circledast \lambda_{i-2} \tag{6.2}$$

for all $g, h \in \mathbf{L}_{i-2}$. This follows directly by taking $\mu = h \circledast \lambda_{i-2} - \lambda_{i-2}$ and $\nu = \lambda_{i-2}$ in (6.1), since by assumption $g \circledast (h \circledast \lambda) = gh \circledast \lambda$ for all $g, h \in \mathbf{L}_{i-2}$ and $\lambda \in \mathfrak{h}_{i-2}$.

The proof that $\circledast$ defines a right action of $\mathbf{R}_i$ on $\mathfrak{h}_i$ is similar and involves showing that

$$(\lambda_{i-2} \circledast g - \lambda_{i-2}) * h + \lambda_{i-2} \circledast h = \lambda_{i-2} \circledast gh \tag{6.3}$$

for all $g, h \in \mathbf{R}_{i-2}$. This identity also follows directly from (6.1) by taking $\mu = \lambda_{i-2} \circledast g - \lambda_{i-2}$ and $\nu = \lambda_{i-2}$. Finally, to show that these actions are compatible it suffices to prove that

$$g * (\lambda_{i-2} \circledast h - \lambda_{i-2}) + g \circledast \lambda_{i-2} = (g \circledast \lambda_{i-2} - \lambda_{i-2}) * h + \lambda_{i-2} \circledast h \tag{6.4}$$

for all $g \in \mathbf{L}_{i-2}$ and $h \in \mathbf{R}_{i-2}$. Since the left hand side is equal to $g \circledast (\lambda_{i-2} \circledast h)$ while the right hand side is equal to $(g \circledast \lambda_{i-2}) \circledast h$, this follows immediately from our inductive hypotheses. Thus $\circledast$ defines compatible left and right group actions of $\mathbf{L}_i$ and $\mathbf{R}_i$ on $\mathfrak{h}_i$.

These results will be of some use later, so we state them in slightly greater generality as the following lemma:

Lemma 6.1. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. Then the following hold:

(1) $g * (\lambda * h) = (g * \lambda) * h$ and $g \circledast (\lambda \circledast h) = (g \circledast \lambda) \circledast h$ for all $g \in \mathbf{L}_i$, $h \in \mathbf{R}_i$, and $\lambda \in \mathfrak{h}_i$.

(2) If $i > 0$ and $X \in \mathfrak{i}_{i-1}$, then $g * X \in i_{i-1}$ and $X * h \in \mathfrak{i}_{i-1}$ for all $g \in \mathbf{L}_i$ and $h \in \mathbf{R}_i$.

(3) $g * (h \circledast \lambda - \lambda) = gh \circledast \lambda - g \circledast \lambda$ for all $g, h \in \mathbf{L}_i$ and $\lambda \in \mathfrak{h}_i$.

(4) $(\lambda \circledast g - \lambda) * h = \lambda \circledast gh - \lambda \circledast h$ for all $g, h \in \mathbf{R}_i$ and $\lambda \in \mathfrak{h}_i$.

(5) $g * (\lambda \circledast h - \lambda) + g \circledast \lambda = (g \circledast \lambda - \lambda) * h + \lambda \circledast h$ for all $g \in \mathbf{L}_i$, $h \in \mathbf{R}_i$, and $\lambda \in \mathfrak{h}_i$.

(6) If $i > 0$, then

$$(g^{-1} \circledast \lambda - \lambda)(\lambda_{i-1} \circledast h - \lambda_{i-1}) = (\lambda \circledast h^{-1} - \lambda)(g \circledast \lambda_{i-1} - \lambda_{i-1})$$

for all $g \in \mathbf{L}_i$, $h \in \mathbf{R}_i$, and $\lambda \in \mathfrak{h}_i$.

Proof. We gave the first two results above, and proved (3)-(5) with i replaced by $i-2$. Since $\circledast$ defines an action of $\mathbf{L}_i$ and $\mathbf{R}_i$ on $\mathfrak{h}_i$, the same proofs now apply to the present case. Observe that (6) is well-defined since by definition $\lambda_{i-1} \circledast h - \lambda_{i-1} \in \mathfrak{i}_{i-1}$ and $g \circledast \lambda_{i-1} - \lambda_{i-1} \in \mathfrak{i}_{i-1}$. To prove this result, we proceed by induction on i. One can check (6) directly for $i = 1$ using (5), so suppose $i > 1$ and the identity holds if we replace i with $i - 1$. Using (5), we then have

$$\begin{aligned}(g^{-1} \circledast \lambda - \lambda)(\lambda_{i-1} \circledast h - \lambda_{i-1}) &= \lambda(g * (\lambda_{i-1} \circledast h - \lambda_{i-1}) - \lambda_{i-1} \circledast h - \lambda_{i-1}) + k_1, \\ &= \lambda((g \circledast \lambda_{i-1} - \lambda_{i-1}) * h - g \circledast \lambda_{i-1} - \lambda_{i-1}) + k_1 \\ &= (\lambda \circledast h^{-1} - \lambda)(g \circledast \lambda_{i-1} - \lambda_{i-1}) + k_1 - k_2\end{aligned}$$

where $k_1 = (\lambda_{i-1} \circledast h - \lambda_{i-1})(g^{-1} \circledast \lambda_{i-2} - \lambda_{i-2})$ and $k_2 = (g \circledast \lambda_{i-1} - \lambda_{i-1})(\lambda_{i-2} \circledast h^{-1} - \lambda_{i-2})$. By assumption $k_1 = k_2$, so by induction (6) holds for all i. □

The last property requiring verification is that the set $\mathfrak{i}_i$ is a vector space over $\mathbb{F}_q$. For this we will prove the following lemma:

Lemma 6.2. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. Then the following hold:

(1) Let $g, h \in \mathbf{L}_i$ and $\tilde{g}, \tilde{h} \in \mathbf{R}_i$. Fix $a, b \in \mathbb{F}_q$, and define $x, \tilde{x} \in \mathbf{G}$ by

$$x = \begin{cases} 1 + a(g - 1) + b(h - 1), & \text{if } i \text{ is even,} \\ (1 + a(g^{-1} - 1) + b(h^{-1} - 1))^{-1}, & \text{if } i \text{ is odd.} \end{cases}$$

$$\tilde{x} = \begin{cases} 1 + a(\tilde{g} - 1) + b(\tilde{h} - 1), & \text{if } i \text{ is even,} \\ (1 + a(\tilde{g}^{-1} - 1) + b(\tilde{h}^{-1} - 1))^{-1}, & \text{if } i \text{ is odd.} \end{cases}$$

Then $x \in \mathbf{L}_i$ and $\tilde{x} \in \mathbf{R}_i$, and

$$\begin{aligned} x \circledast \lambda &= (1 - a - b)\lambda + a(g \circledast \lambda) + b(h \circledast \lambda), \\ \lambda \circledast \tilde{x} &= (1 - a - b)\lambda + a(\lambda \circledast \tilde{g}) + b(\lambda \circledast \tilde{h}). \end{aligned}$$

(2) $\mathbf{L}_i$ and $\mathbf{R}_i$ are algebra groups for all i.

(3) Let $\mathfrak{l}_i = \mathbf{L}_i \circledast \lambda_i - \lambda_i$ and $\mathfrak{r}_i = \lambda_i \circledast \mathbf{R}_i - \lambda_i$ for $i = 0, 1, \ldots, n$. Then $\mathfrak{l}_i$ and $\mathfrak{r}_i$ are $\mathbb{F}_q$-subspaces of $\mathfrak{h}_i$ for all i.

(4) $\mathfrak{i}_i$ is an $\mathbb{F}_q$-subspace of $\mathfrak{h}_i$ for all i.

Proof. Since by definition $\mathfrak{i}_i = \mathfrak{l}_i \cap \mathfrak{r}_i$, these results imply that $\mathfrak{i}_i$ is a subspace, and therefore by induction that our concept of a decomposition sequence is well-defined. As we have not verified this directly, we must proceed by induction on i.

For $i = 0$ the left and right $\circledast$ actions of $\mathbf{L}_0 = \mathbf{R}_0 = \mathbf{G}$ on $\mathfrak{h}_0 = \mathfrak{n}$ are transitive, so $\mathfrak{r}_0 = \mathfrak{l}_0 = \mathfrak{h}_0$ are $\mathbb{F}_q$-subspaces and $\mathbf{L}_0$ and $\mathbf{R}_0$ are algebra groups. In addition, if $g, h \in \mathbf{L}_0$, $\tilde{g}, \tilde{h} \in \mathbf{R}_0$, and $\lambda \in \mathfrak{h}_0$, then for any $a, b \in \mathbb{F}_q$, one can check directly that

$$(1 + a(g-1) + b(h-1)) \circledast \lambda = (1-a-b)\lambda + a(g \circledast \lambda) + b(h \circledast \lambda),$$
$$\lambda \circledast (1 + a(\tilde{g}-1) + b(\tilde{h}-1)) = (1-a-b)\lambda + a(\lambda \circledast \tilde{g}) + b(\lambda \circledast \tilde{h}).$$

Since $\mathfrak{i}_0 = \mathfrak{n}$, $\mathbf{L}_1 = \mathbf{R}_1 = \mathbf{G}$ are again algebra groups, and if $g, h \in \mathbf{L}_1$, $\tilde{g}, \tilde{h} \in \mathbf{R}_1$, and $\lambda \in \mathfrak{h}_1$, then for any $a, b \in \mathbb{F}_q$, one can check that

$$(1 + a(g^{-1}-1) + b(h^{-1}-1))^{-1} \circledast \lambda = (1-a-b)\lambda + a(g \circledast \lambda) + b(h \circledast \lambda),$$
$$\lambda \circledast (1 + a(\tilde{g}^{-1}-1) + b(\tilde{h}^{-1}-1))^{-1} = (1-a-b)\lambda + a(\lambda \circledast \tilde{g}) + b(\lambda \circledast \tilde{h}).$$

Now let $i > 1$ and suppose for $0 \leq j < i$, the sets $\mathbf{L}_j, \mathbf{R}_j$ are groups, the operations $*, \circledast$ define compatible left and right group actions of $\mathbf{L}_j$ and $\mathbf{R}_j$ on $\mathfrak{h}_j$, the set $\mathfrak{i}_j$ is a vector space, and the results of the lemma hold. It is a simple inductive exercise to show from here that (1) holds for $j = i$; we omit this proof for the sake of brevity. Now let $g, h \in \mathbf{L}_i$ so that for some $\tilde{g}, \tilde{h} \in \mathbf{R}_i$, $g \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{g}$ and $h \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{h}$. Fix $a, b \in \mathbb{F}_q$ and define $x, \tilde{x} \in \mathbf{G}$ as in (1). Then $x \in \mathbf{L}_{i-1}$ and $\tilde{x} \in \mathbf{R}_{i-1}$ since by assumption $\mathbf{L}_{i-1}$ and $\mathbf{R}_{i-1}$ are algebra groups, and by (1) we have that $x \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{x}$. In the even case this suffices to show that $\mathbf{L}_i - 1$ and $\mathbf{R}_i - 1$ are $\mathbb{F}_q$-vector spaces and therefore associative nilpotent $\mathbb{F}_q$-algebras. In the odd case this result still follows by replacing $g, h, \tilde{g}, \tilde{h}$ with their inverses and noting that $x \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{x}$ implies $x^{-1} \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{x}^{-1}$. Thus $\mathbf{L}_i$ and $\mathbf{R}_i$ are algebra groups. In addition $\mathfrak{l}_i$ is a subspace, since if $\gamma_1 = g \circledast \lambda_i - \lambda_i \in \mathfrak{l}_i$ and $\gamma_2 = h \circledast \lambda_i - \lambda_i \in \mathfrak{l}_i$, then $a\gamma_1 + b\gamma_2 = x \circledast \lambda \in \mathfrak{l}_i$. By the same argument, it follows that $\mathfrak{r}_i$ is also a subspace. (4) then follows immediately since $\mathfrak{i}_i = \mathfrak{l}_i \cap \mathfrak{r}_i$, and so by induction the results of the lemma hold for all i, and the concept of a decomposition sequence is well-defined. $\square$

All of our definitions thus far have been completely symmetric with respect to left and right. The following lemma describes some consequences of this symmetry which will be useful in the next section. In particular, the lemma introduces a notation for the $\circledast$-stabilizer subgroups of $\mathbf{L}_i$ and $\mathbf{R}_i$ with respect to a functional $\lambda \in \mathfrak{h}_i$. These subgroups will be especially important in defining the characters indexed by a decomposition sequence. In particular, the characters in question will be induced from linear characters of the stabilizer subgroups.

Lemma 6.3. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and $\lambda \in \mathfrak{h}_i$, let

$$\mathbf{LStab}_S(\lambda) = \{g \in \mathbf{L}_i \mid g \circledast_i \lambda = \lambda\} \qquad \text{and} \qquad \mathbf{RStab}_S(\lambda) = \{g \in \mathbf{R}_i \mid \lambda \circledast_i g = \lambda\}.$$

The following then hold:

(1) $\mathbf{LStab}_S(\lambda)$ and $\mathbf{RStab}_S(\lambda)$ are algebra groups for all $\lambda \in \mathfrak{h}_i$, and so the orbit sizes $|\mathbf{L}_i \circledast \lambda| = |\mathbf{L}_i|/|\mathbf{LStab}_S(\lambda)|$ and $|\lambda \circledast \mathbf{R}_i| = |\mathbf{R}_i|/|\mathbf{RStab}_S(\lambda)|$ are powers of q.

(2) If $\lambda \in \mathfrak{h}_i$ and $i > 0$, then

$$\mathbf{LStab}_S(\lambda) = \{g \in \mathbf{L}_i \mid g \circledast_{i-1} \lambda_{i-1} - \lambda_{i-1} \subseteq \ker(\mu - \lambda) \text{ for all } \mu \in \lambda \circledast_i \mathbf{R}_i\},$$
$$\mathbf{RStab}_S(\lambda) = \{h \in \mathbf{R}_i \mid \lambda_{i-1} \circledast_{i-1} h - \lambda_{i-1} \subseteq \ker(\mu - \lambda) \text{ for all } \mu \in \mathbf{L}_i \circledast_i \lambda\}.$$

(3) $\mathbf{LStab}_S(\lambda_{i-1}) \lhd \mathbf{L}_i$ and $\mathbf{RStab}_S(\lambda_{i-1}) \lhd \mathbf{R}_i$ for all $i > 0$.

(4) $\mathbf{LStab}_S(\lambda_{i-1}) \lhd \mathbf{LStab}_S(\lambda)$ and $\mathbf{RStab}_S(\lambda_{i-1}) \lhd \mathbf{LStab}_S(\lambda)$ for all $\lambda \in \mathfrak{h}_i$ and $i > 0$.

(5) Let $\mathbf{QL}_i = \mathbf{L}_i/\mathbf{LStab}_S(\lambda_{i-1})$ and $\mathbf{QR}_i = \mathbf{R}_i/\mathbf{RStab}_S(\lambda_{i-1})$ for $i > 0$. Then the quotient groups are isomorphic: $\mathbf{QL}_i \cong \mathbf{QR}_i$.

(6) $|\mathbf{L}_i \circledast \lambda| = |\lambda \circledast \mathbf{R}_i|$ and $|\mathbf{LStab}_S(\lambda)| = |\mathbf{RStab}_S(\lambda)|$ for all $\lambda \in \mathfrak{h}_i$ and $i \geq 0$.

Proof. (1) follows immediately from the first result in the preceding lemma. To show (2), let $g \in \mathbf{L}_i$ and $h \in \mathbf{R}_i$ and observe by Lemma 6.1 that

$$(g^{-1} \circledast \lambda - \lambda)(\lambda_{i-1} \circledast h^{-1} - \lambda_{i-1}) = (\lambda \circledast h - \lambda)(g \circledast \lambda_{i-1} - \lambda_{i-1}).$$

Since every $X \in \mathfrak{i}_{i-1}$ is of the form $\lambda_{i-1} \circledast h^{-1} - \lambda_{i-1}$ for some $h \in \mathbf{R}_i$ and since $g \circledast \lambda = \lambda$ iff $g^{-1} \circledast \lambda - \lambda = 0$, it follows that $g \in \mathbf{LStab}_S(\lambda)$ iff $g \circledast \lambda_{i-1} - \lambda_{i-1} \subseteq \ker(\lambda \circledast h - \lambda)$ for all $h \in \mathbf{R}_i$, or equivalently iff $g \circledast \lambda_{i-1} - \lambda_{i-1} \subseteq \ker(\mu - \lambda)$ for all $\mu \in \lambda \circledast \mathbf{R}_i$. The proof of the second half of (2) is similar.

To prove (3), observe that if $g \in \mathbf{LStab}_S(\lambda_{i-1})$ and $x \in \mathbf{L}_i$ so that $x \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{x}$ for some $\tilde{x} \in \mathbf{R}_i$, then $x^{-1}gx \circledast \lambda_{i-1} = x^{-1}g \circledast \lambda_{i-1} \circledast \tilde{x} = x^{-1} \circledast \lambda_{i-1} \circledast \tilde{x} = x^{-1}x \circledast \lambda_{i-1} = \lambda_{i-1}$, so $x^{-1}gx \in \mathbf{LStab}_S(\lambda_{i-1})$. Therefore $\mathbf{LStab}_S(\lambda_{i-1}) \lhd \mathbf{L}_i$, and the proof that $\mathbf{RStab}_S(\lambda_{i-1}) \lhd \mathbf{R}_i$ is similar. For (4), it suffices to show that $\mathbf{LStab}_S(\lambda_{i-1}) \subseteq \mathbf{LStab}_S(\lambda)$ and $\mathbf{RStab}_S(\lambda_{i-1}) \subseteq \mathbf{RStab}_S(\lambda)$ for all $\lambda \in \mathfrak{h}_i$. Since every $X \in \mathfrak{i}_{i-1}$ is of the form $X = \lambda_{i-1} \circledast h - \lambda_{i-1}$ for some $h \in \mathbf{R}_i$, if $g \in \mathbf{LStab}_S(\lambda_{i-1})$ and $\lambda \in \mathfrak{h}_i$, then by Lemma 6.1,

$$(g \circledast \lambda - \lambda)(X) = (\lambda \circledast h^{-1} - \lambda)(g^{-1} \circledast \lambda_{i-1} - \lambda_{i-1}) = (\lambda \circledast h^{-1} - \lambda)(\lambda_{i-1} - \lambda_{i-1}) = 0.$$

Since this holds for all $X \in \mathfrak{i}_i$, we have $g \circledast \lambda = \lambda$ so $\mathbf{LStab}_S(\lambda_{i-1}) \subseteq \mathbf{LStab}_S(\lambda)$. A similar argument shows that $\mathbf{RStab}_S(\lambda_{i-1}) \subseteq \mathbf{RStab}_S(\lambda)$.

To prove (5) consider the map $\mathbf{QL}_i \to \mathbf{QR}_i$ defined on coset representatives by $g \mapsto \tilde{g}$ where $g \in \mathbf{L}_i$ and $\tilde{g} \in \mathbf{R}_i$ satisfy $g \circledast \lambda_{i-1} = \lambda_{i-1} \circledast \tilde{g}$. It is a simple exercise to show that this map is well-defined and gives an isomorphism between the two quotient groups. Finally, for (6) we proceed by induction on i. Since $\mathbf{L}_0 \circledast \lambda = \lambda \circledast \mathbf{R}_0 = \mathfrak{h}_0 = \mathfrak{n}$ and $\mathbf{LStab}_S(\lambda) = \mathbf{RStab}_S(\lambda) = \{1\}$ for all $\lambda \in \mathfrak{h}_0$, our result holds automatically for $i = 0$. Suppose $i > 0$ and $|\mathbf{L}_i \circledast \lambda_i| = |\lambda_i \circledast \mathbf{R}_i|$ and $|\mathbf{LStab}_S(\lambda_i)| = |\mathbf{RStab}_S(\lambda_i)|$. Note that this hypothesis combined with (5) implies $|\mathbf{L}_{i+1}| = |\mathbf{R}_{i+1}|$. Now observe that if $\lambda \in \mathfrak{h}_{i+1}$, then by (2) and (4) we have

$$\frac{|\mathbf{LStab}_S(\lambda)|}{|\mathbf{LStab}_S(\lambda_i)|} = \frac{|\mathfrak{i}_i|}{|\lambda \circledast \mathbf{R}_{i+1}|} = \frac{|\mathbf{R}_{i+1}|/|\mathbf{RStab}_S(\lambda_i)}{|\mathbf{R}_{i+1}|/|\mathbf{RStab}_S(\lambda)|} = \frac{|\mathbf{RStab}_S(\lambda)|}{|\mathbf{RStab}_S(\lambda_i)|},$$

so $|\mathbf{LStab}_S(\lambda)| = |\mathbf{RStab}_S(\lambda)|$ and $|\mathbf{L}_{i+1} \circledast \lambda| = \frac{|\mathbf{L}_{i+1}|}{|\mathbf{LStab}_S(\lambda)|} = \frac{|\mathbf{R}_{i+1}|}{|\mathbf{RStab}_S(\lambda)|} = |\lambda \circledast \mathbf{R}_{i+1}|$. Hence (6) holds for all i and the proof of the lemma is complete. □

The preceding lemma gives rise to the following corollary:

Corollary 6.1. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. Write $n_{S,i} = \frac{|\mathbf{L}_i \circledast \lambda_i|}{|\mathfrak{i}_i|} = \frac{|\lambda_i \circledast \mathbf{R}_i|}{|\mathfrak{i}_i|}$. Then the two-sided orbit $\mathbf{L}_i \circledast \lambda_i \circledast \mathbf{R}_i$ is a union of $n_{S,i}$ left orbits that are transitively permuted by the right $\circledast$-action of $\mathbf{R}_i$, and it is also the union of $n_{S,i}$ right orbits that are transitively permuted by the left $\circledast$-action of $\mathbf{L}_i$. Furthermore, $|\mathbf{L}_i \circledast \lambda_i \circledast \mathbf{R}_i|$ and $n_{S,i}$ are powers of q.

Proof. Since $|\mathbf{L}_i \circledast \lambda_i|$, $|\lambda_i \circledast \mathbf{R}_i|$, and $|\mathfrak{i}_i|$ are powers of q by the preceding lemma and the fact that $\mathfrak{i}_i$ is a vector space over $\mathbb{F}_q$, the results in this corollary follow immediately from Lemma 4.1. □

7 Module constructions

Now that we have introduced the definition of a decomposition sequence, we show how to use such sequences to decompose the regular representation of an algebra group. As in Section 4, fix a nontrivial homomorphism $\theta : \mathbb{F}_q^\times \to \mathbb{C}^\times$ from the additive group of $\mathbb{F}_q$ to the multiplicative group of the complex numbers $\mathbb{C}$. In other words, let θ be any one of the $q-1$ non-prinicipal linear characters of the additive group $\mathbb{F}_q^+$.

Given an element $w = \sum_{g \in \mathbf{G}} c_g g \in \mathbb{C}\mathbf{G}$ where each $c_g \in \mathbb{C}$, let $\overline{w} = \sum_{g \in \mathbf{G}} \overline{c_g} g$. The usual properties of complex conjugation then carry over to the group algebra, as clearly $\overline{x} \cdot \overline{y} = \overline{xy}$ and $\overline{x} + \overline{y}$ for all $\overline{x+y}$ for all $x, y \in \mathbb{C}\mathbf{G}$. We now have the following definition:

Definition 2. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ of an algebra group $\mathbf{G} = 1 + \mathfrak{n}$, let $v_{S,i} : \mathfrak{h}_i \to \mathbb{C}\mathbf{G}$ be the map defined recursively by

$$\begin{aligned} v_{S,0}(\lambda) &= \lambda + 1 \in \mathbf{G}, && \text{for } \lambda \in \mathfrak{h}_0, \\ v_{S,i+1}(\lambda) &= \frac{1}{|\mathfrak{i}_i|} \sum_{X \in \mathfrak{i}_i} \theta(\lambda(X)) \overline{v_{S,i}(\lambda_i + X)}, && \text{for } \lambda \in \mathfrak{h}_{i+1},\ 0 \le i < n. \end{aligned}$$

To simplify notation, we will typically omit the second subscript and abbreviate v_S when the context is clear.

Observe that $v_{S,1}(\lambda) = v_\lambda$ for $\lambda \in \mathfrak{n}^*$ in the notation of Section 4. Thus the following definition should look familiar for the case $n = 1$:

Definition 3. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ of an algebra group $\mathbf{G}$, let V_S and M_S be the $\mathbb{C}\mathbf{G}$-modules

$$V_S = \mathbb{C}\mathbf{G} v_S(\lambda_n) \qquad \text{and} \qquad M_S = \mathbb{C}\mathbf{G} v_S(\lambda_n) \mathbb{C}\mathbf{G},$$

and let χ_S be the character of V_S.

Since $v_S(\lambda) = \lambda + 1$ for all $\lambda \in \mathfrak{h}_0 = \mathfrak{n}$, we have $V_S = M_S = \mathbb{C}\mathbf{G}$ if $S \in \mathscr{D}(\mathbf{G})$ has a single term. More significantly, comparing this definition to equation (4.2), we see that $V_S = V^{\lambda_1}$ is the module of a supercharacter if S has only two terms. Thus the characters χ_S for two-term decomposition sequences are just the supercharacters of $\mathbf{G}$. For $n > 1$, we can say little at first glance about the modules V_S and characters χ_S. However, in the course of this section, we will show that these generalized characters retain many of the useful properties of supercharacters.

For the duration of this section, fix an algebra group $\mathbf{G} = 1 + \mathfrak{n}$ and a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$. As our first result, we show how to multiply the vectors $v_{S,i}(\lambda)$ by elements of $\mathbf{L}_i$ and $\mathbf{R}_i$. We begin with the following definitions:

Definition 4. Given $0 \le i \le n$, let $\alpha_{S,i} : \mathbf{L}_i \times \mathfrak{h}_i \to \mathbb{C}^\times$ and $\beta_{S,i} : \mathfrak{h}_i \times \mathbf{R}_i \to \mathbb{C}^\times$ be the maps defined recursively by

$$\begin{aligned} \alpha_{S,0}(g, \lambda) &= 1, && g \in \mathbf{L}_0,\ \lambda \in \mathfrak{h}_0, \\ \alpha_{S,i+1}(g, \lambda) &= \theta\left((g \circledast \lambda)(\lambda_i - g \circledast \lambda_i)\right) \overline{\alpha_{S,i}(g, \lambda_i)}, && g \in \mathbf{L}_{i+1},\ \lambda \in \mathfrak{h}_{i+1},\ 0 \le i < n, \end{aligned}$$

$$\begin{aligned} \beta_{S,0}(\lambda, h) &= 1, && h \in \mathbf{R}_0,\ \lambda \in \mathfrak{h}_0, \\ \beta_{S,i+1}(\lambda, h) &= \theta\left((\lambda \circledast h)(\lambda_i - \lambda_i \circledast h)\right) \overline{\beta_{S,i}(\lambda_i, h)}, && h \in \mathbf{R}_{i+1},\ \lambda \in \mathfrak{h}_{i+1},\ 0 \le i < n. \end{aligned}$$

for $g \in \mathbf{L}_i$, $\lambda \in \mathfrak{h}_i$, and $h \in \mathbf{R}_i$. As before, to simplify notation, we will typically omit the second subscripts and write α_S and β_S when the context is clear.

We now have the following instrumental result:

Lemma 7.1. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and $\lambda \in \mathfrak{h}_i$ for some $0 \le i \le n$. Then the following hold:

(1) $g v_S(\lambda) = \alpha_S(g, \lambda) v_S(g \circledast \lambda)$ for all $g \in \mathbf{L}_i$.

(2) $v_S(\lambda) h = \beta_S(\lambda, h) v_S(\lambda \circledast h)$ for all $h \in \mathbf{R}_i$.

Proof. The proofs of these identities are essentially identical, so we only show that $g v_S(\lambda) = \alpha_S(g, \lambda) v_S(g \circledast \lambda)$. As usual, we proceed by induction on i. The lemma holds trivially for $i = 0$, so suppose the lemma also holds for some $i > 0$. For $g \in \mathbf{L}_{i+1}$ and $\lambda \in \mathfrak{h}_{i+1}$, we then have

$$g v_S(\lambda) = \frac{1}{|\mathfrak{i}_i|} \sum_{X \in \mathfrak{i}_i} \theta\left(\lambda(X)\right) \overline{\alpha_{S,i}(g, \lambda_i + X) v_S(\lambda_i + g * X + g \circledast \lambda_i - \lambda_i)},$$

by Lemma 6.1. Since by definition $g \circledast \lambda_i - \lambda_i \in \mathfrak{i}_i$, the map $X \mapsto g * X + g \circledast \lambda_i - \lambda_i$ permutes the vector space $\mathfrak{i}_i$. The inverse of this map is given by the transformation $X \mapsto g^{-1} * X + g^{-1} \circledast \lambda_i - \lambda_i$, so by making the substitution $X \mapsto g * X + g \circledast \lambda_i - \lambda_i$ in the preceding expression we obtain

$$gv_S(\lambda) = \frac{1}{|\mathfrak{i}_i|} \sum_{X \in \mathfrak{i}_i} \theta\left(\lambda(g^{-1} * X + g^{-1} \circledast \lambda_i - \lambda_i)\right) \overline{\alpha_{S,i}(g, g^{-1} * X + g^{-1} \circledast \lambda_i) v_S(\lambda_i + X)}.$$

Since

$$\begin{aligned}\overline{\alpha_{S,i}(g, g^{-1} * X + g^{-1} \circledast \lambda_i)} &= \alpha_{S,i-1}(g, \lambda_{i-1}) \overline{\theta\left((g \circledast (g^{-1} * X + g^{-1} \circledast \lambda_i))(\lambda_{i-1} - g \circledast \lambda_{i-1})\right)} \\ &= \alpha_{S,i-1}(g, \lambda_{i-1}) \theta\left(\Phi^{\mathbf{L}}_{S,i+1}[g](X + \lambda_i)\right)\end{aligned}$$

and $\lambda(g^{-1} \circledast \lambda_i - \lambda_i) = (g \circledast \lambda)(\lambda_i - g \circledast \lambda_i) + \Phi^{\mathbf{L}}_{S,i+1}[g](g \circledast \lambda_i - \lambda_i)$, this becomes

$$gv_S(\lambda) = \frac{c}{|\mathfrak{i}_i|} \sum_{X \in \mathfrak{i}_i} \theta\left((g \circledast \lambda)(X)\right) \overline{v(\lambda_i + X)} = cv_S(g \circledast \lambda),$$

where $c = \theta((g \circledast \lambda)(\lambda_i - g \circledast \lambda_i))\theta\left(\Phi^{\mathbf{L}}_{S,i+1}[g](g \circledast \lambda_i)\right) \alpha_{S,i-1}(g, \lambda_{i-1}) = \alpha_{S,i+1}(g, \lambda)$. Thus (1) holds for $i+1$, and hence for all i by induction. $\square$

This result combined with Lemma 6.3 shows that for each $\lambda \in \mathfrak{h}_j$, the vector $v_S(\lambda) = v_{S,j}(\lambda) \in \mathbb{C}\mathbf{G}$ is a simultaneous left and right eigenvector for each $g \in \mathbf{LStab}_S(\lambda_i)$ and $h \in \mathbf{RStab}_S(\lambda_i)$ for all $i < j$. From here, we can almost say what the dimension of V_S is, as the next lemma clarifies.

Lemma 7.2. For any decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$,

$$\dim(V_S) \le \frac{|\mathbf{G}|}{|\mathbf{LStab}_S(\lambda_n)|} = |\mathbf{G}| \prod_{i=0}^{n} \frac{|\mathbf{L}_i \circledast \lambda_i|}{|\mathfrak{h}_i|}.$$

Proof. Let $\widetilde{\mathbf{G}} \subseteq \mathbf{G}$ be a set of coset representatives of $\mathbf{G}/\mathbf{LStab}_S(\lambda_n)$. Then every element of $\mathbf{G}$ can be written $g = xy$ where $x \in \widetilde{\mathbf{G}}$ and $y \in \mathbf{LStab}_S(\lambda_n)$, so V_S is spanned by elements of the form $gv_S(\lambda_n) = \alpha_S(y, \lambda_n) x v_S(\lambda_n)$. Hence $\dim(V_S) \le \frac{|\mathbf{G}|}{|\mathbf{LStab}_S(\lambda_n)|}$. To complete the proof of the lemma, we observe that

$$\prod_{i=0}^{n} \frac{|\mathfrak{h}_i|}{|\mathbf{L}_i \circledast \lambda_i|} = \prod_{i=1}^{n} \frac{|\mathfrak{h}_i|}{|\mathbf{L}_i|/|\mathbf{LStab}_S(\lambda_{i-1})|} \frac{|\mathbf{LStab}_S(\lambda_i)|}{|\mathbf{LStab}_S(\lambda_{i-1})|} = |\mathbf{LStab}_S(\lambda_n)|,$$

where the first equality follows from the fact that $\frac{|\mathfrak{h}_0|}{|\mathbf{L}_0 \circledast \lambda_0|} = 1$, the second equality follows from the fact that $|\mathbf{L}_i|/|\mathbf{LStab}_S(\lambda_{i-1})| = |\mathfrak{i}_{i-1}| = |\mathfrak{h}_i|$, and the third equality follows from the fact that $|\mathbf{LStab}_S(\lambda_0)| = 1$. $\square$

With these preliminary lemmas, we can now answer some more substantial questions about how the modules V_S decompose the group algebra $\mathbb{C}\mathbf{G}$. In order to state these results precisely, we require some further notation to describe the set of decomposition sequences which can be formed by adding a single term to a given sequence S.

Since all decomposition sequences begin with the same first term, the set of decomposition sequences of an algebra group naturally forms a rooted tree, where the children of each sequence $S \in \mathscr{D}(\mathbf{G})$ are simply the decomposition sequences which can be formed by appending a single term $(\lambda, \mathfrak{h}, \mathbf{L}, \mathbf{R}, \mathfrak{i})$ to S. In order to refer to such constructions, we observe that each child of a sequence S is uniquely determined according to Definition 1 by a linear functional $\lambda \in \mathfrak{i}_n^*$. Therefore we adopt the following notation:

Notation. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{R}_i, \mathbf{L}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$, let $\mathfrak{h}_S = \mathfrak{i}_n^*$ denote the dual space of $\mathfrak{i}_n$ and let $\mathbf{L}_S \subseteq \mathbf{L}_n$ and $\mathbf{R}_S \subseteq \mathbf{R}_n$ denote the subgroups defined by

$$\mathbf{L}_S = \{g \in \mathbf{L}_n \mid g \circledast_n \lambda_n - \lambda_n \in \mathfrak{i}_n\},$$
$$\mathbf{R}_S = \{g \in \mathbf{R}_n \mid \lambda_n \circledast_n g - \lambda_n \in \mathfrak{i}_n\}.$$

$\mathbf{L}_S$ and $\mathbf{R}_S$ act on $\mathfrak{h}_S$ on the left and right by $*$ and $\circledast$ as defined as in Definition 1. Now, for any $\lambda \in \mathfrak{h}_S$, let $\mathfrak{i}_\lambda \subseteq \mathfrak{h}_S$ denote the subspace $\mathfrak{i}_\lambda = \mathbf{L}_S \circledast \lambda \cap \lambda \circledast \mathbf{R}_S - \lambda$, and define $S|\lambda \in \mathscr{D}(\mathbf{G})$ as the decomposition sequence formed by appending the final term $(\lambda, \mathfrak{h}_S, \mathbf{L}_S, \mathbf{R}_S, \mathfrak{i}_\lambda)$ to S.

Given two sequences $S, T \in \mathscr{D}(\mathbf{G})$, we say that T is a *child* of S if $T = S|\lambda$ for some $\lambda \in \mathfrak{h}_S$. Likewise, we say that T is a *descendent* of S if there are sequences $S = S_0, S_1, \dots, S_k = T \in \mathscr{D}(\mathbf{G})$ such that S_i is a child of S_{i-1} for each $i = 1, \dots, k$. We naturally extend these relations from decomposition sequences to the characters they index. With this these additional definitions, we can now begin to describe when two decomposition sequences index the same $\mathbb{C}\mathbf{G}$-module.

Lemma 7.3. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{R}_i, \mathbf{L}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. If $\lambda, \gamma \in \mathfrak{h}_S$, then the following hold:

(1) If $\gamma \in \mathbf{L}_S \circledast \lambda$, then $V_{S|\lambda} = V_{S|\gamma}$ and $M_{S|\lambda} = M_{S|\gamma}$.

(2) If $\gamma \in \lambda \circledast \mathbf{R}_S$, then $V_{S|\lambda} \cong V_{S|\gamma}$ and $M_{S|\lambda} = M_{S|\gamma}$.

Proof. To show (1), we note that if $\gamma \in \mathbf{L}_S \circledast \lambda$ then by the preceding lemma $V_{S|\gamma} = gV_{S|\gamma} \subseteq V_{S|\lambda} = g^{-1}V_{S|\lambda} \subseteq V_{S|\gamma}$ for some $g \in \mathbf{L}_S$ so we have equality throughout. For (2), we observe that for any $\mathbf{G}$-module $V \subseteq \mathbb{C}\mathbf{G}$, the map $V \to Vg = \{xg \mid x \in V\}$ given by $x \mapsto xg$ for some $g \in \mathbf{G}$ is a module isomorphism by the associativity of multiplication in $\mathbf{G}$. If $\gamma \in \lambda \circledast \mathbf{R}_S$, then by the preceding lemma $V_{S|\gamma} = V_{S|\lambda}g \cong V_{S|\lambda}$ for some $g \in \mathbf{R}_S$. $\square$

This lemma suggests a natural way of decomposing V_S into submodules by choosing a set of descendants of S. In particular, we have the following theorem:

Notation. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$, let m_S denote the positive integer defined by

$$m_S = \frac{|\mathbf{G}|/|\mathfrak{i}_n|}{|\mathbf{LStab}_S(\lambda_n)|} = \prod_{i=0}^n \frac{|\mathbf{L}_i \circledast \lambda_i \circledast \mathbf{R}_i|}{|\mathbf{L}_i \circledast \lambda_i|}.$$

Theorem 7.1. Fix a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{R}_i, \mathbf{L}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and let $\mathscr{I} \subseteq \mathfrak{h}_S$ be a set of representatives of the two-sided orbits of $\mathfrak{h}_S$ under the $\circledast$ actions of $\mathbf{L}_S$ and $\mathbf{R}_S$. Then

$$\chi_S = \frac{1}{m_S} \sum_{\lambda \in \mathscr{I}} m_{S|\lambda} \chi_{S|\lambda}.$$

Proof. If $n = 0$, then $V_S = \mathbb{C}\mathbf{G}$ so $\dim(V_S) = |\mathbf{G}|$ and we have equality in Lemma 7.2. Now suppose $n > 0$ and we still have equality in Lemma 7.2. In this case, if $\lambda \in \mathfrak{h}_S$ then the lemma gives $\dim(V_{S|\lambda}) \leq \frac{|\mathbf{L}_S \circledast \lambda|}{|\mathfrak{h}_S|} \dim(V_S)$. With slight abuse of notation, write $v = v_{S|\lambda}$ for all $\lambda \in \mathfrak{h}_S$, and let $u_\lambda = \sum_{\gamma \in \mathbf{L}_S \circledast \lambda} v(\gamma) \in V_{S|\lambda}$. Let $\mathscr{I}_{\mathrm{L}} \subseteq \mathfrak{h}_S$ be a set of representatives of the left orbits of $\mathfrak{h}_S$ under the $\circledast$ action of $\mathbf{L}_S$, and consider the sum

$$\sum_{\lambda \in \mathscr{I}_{\mathrm{L}}} u_\lambda = \sum_{\lambda \in \mathfrak{h}_S} v(\lambda) = \frac{1}{|\mathfrak{i}_n|} \sum_{X \in \mathfrak{i}_n} \overline{v_S(\lambda_n + X)} \sum_{\lambda \in \mathfrak{h}_S} \theta(\lambda(X)).$$

If $X \neq 0$, then the map $\mathfrak{h}_S \to \mathbb{C}^\times$ given by $\lambda \mapsto \theta(\lambda(X))$ is a nontrivial homomorphism so $\sum_{\lambda \in \mathfrak{h}_S} \theta(\lambda(X)) = 0$, while if $X = 0$ then $\sum_{\lambda \in \mathfrak{h}_S} \theta(\lambda(X)) = |\mathfrak{h}_S| = |\mathfrak{i}_n|$. Thus $\sum_{\lambda \in \mathscr{I}_{\mathrm{L}}} u_\lambda = v_S(\lambda_n)$, so since $V_S = \mathbb{C}\mathbf{G}v(\lambda_n)$, it follows that $V_S = \sum_{\lambda \in \mathscr{I}_{\mathrm{L}}} V_{S|\lambda}$. Here we use $\sum$ instead of

$\bigoplus$ to distinguish between a sum and direct sum. Since the characters of isomorphic modules are the same, it then follows by assumption that

$$\dim(V_S) = \dim\left(\sum_{\lambda\in\mathscr{I}_{\mathrm{L}}} V_{S|\lambda}\right) \le \sum_{\lambda\in\mathscr{I}_{\mathrm{L}}} \dim(V_{S|\lambda}) \le \sum_{\lambda\in\mathscr{I}_{\mathrm{L}}} \frac{|\mathbf{L}_S \circledast \lambda|}{|\mathfrak{h}_S|}\dim(V_S) = \dim(V_S).$$

All inequalities in this statement must therefore become equalities, which necessarily gives $V_S = \bigoplus_{\lambda\in\mathscr{I}_{\mathrm{L}}} V_{S|\lambda}$. Since $V_{S|\lambda} \cong V_{S|\gamma}$ if $\gamma \in \lambda \circledast \mathbf{R}_S$, and since there are exactly $\frac{m_{S|\lambda}}{m_S} = \frac{|\mathbf{L}_S\circledast\lambda\circledast\mathbf{R}_S|}{|\mathbf{L}_S\circledast\lambda|}$ distinct right orbits in the two-sided orbit $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$, it follows that $V_S = \bigoplus_{\lambda\in\mathscr{I}} (V_{S|\lambda})^{\oplus m_{S|\lambda}/m_S}$ and $\chi_S = \frac{1}{m_S}\sum_{\lambda\in\mathscr{I}} m_{S|\lambda}\chi_{S|\lambda}$. This implies as a corollary that $\dim(V_{S|\lambda}) = \frac{|\mathbf{L}_S\circledast\lambda|}{|\mathfrak{h}_S|}\dim(V_S)$ for all $\lambda \in \mathfrak{h}_S$, so we have equality in Lemma 7.2. This proves by induction that we have equality in Lemma 7.2 for all decomposition sequences, and in turn that this lemma holds for all sequences. □

The following properties are immediate from the theorem:

Theorem 7.2. For any decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$, the following hold:

(1) $\dim(V_S) = \dfrac{|\mathbf{G}|}{|\mathbf{LStab}_S(\lambda_n)|} = |\mathbf{G}|\displaystyle\prod_{i=0}^{n}\frac{|\mathbf{L}_i \circledast \lambda_i|}{|\mathfrak{h}_i|}$.

(2) A basis for V_S is given by the set $\left\{g_1 \dots g_{n-1} v_S(\lambda) \mid g_i \in \widetilde{\mathbf{L}}_i,\ \lambda \in \mathbf{L}_n \circledast \lambda_n\right\}$, where $\widetilde{\mathbf{L}}_i$ denotes a fixed set of representatives of the cosets $\mathbf{L}_i/\mathbf{L}_{i+1}$ for $0 < i < n$.

(3) If $\lambda, \gamma \in \mathfrak{h}_S$ and $\gamma \notin \mathbf{L}_S \circledast \lambda$, then $V_{S|\lambda} \cap V_{S|\gamma} = \{0\}$.

Proof. (1) was proved is the theorem. Given (1), it follows from Lemma 7.2 that the set $\{xv_S(\lambda_n) \mid x \in \widetilde{\mathbf{G}}\}$ gives a basis for V_S, where $\widetilde{\mathbf{G}} \subseteq \mathbf{G}$ is a set of coset representatives of $\mathbf{G}/\mathbf{LStab}_S(\lambda_n)$. Since we have a descending sequence of subgroups $\mathbf{G} = \mathbf{L}_0 \supseteq \mathbf{L}_1 \supseteq \cdots \supseteq \mathbf{L}_n \supseteq \mathbf{LStab}_S(\lambda_n)$, we can write $\widetilde{\mathbf{G}}$ as the product $\widetilde{\mathbf{G}} = \widetilde{\mathbf{L}}_0\widetilde{\mathbf{L}}_1 \dots \widetilde{\mathbf{L}}_n$ where $\widetilde{\mathbf{L}}_i$ is a set of coset representatives of $\mathbf{L}_i/\mathbf{L}_{i+1}$ for $0 \le i < n$ and $\widetilde{\mathbf{L}}_n$ is a set of coset representatives of $\mathbf{L}_n/\mathbf{LStab}_S(\lambda_n)$. Since $\mathbf{L}_0 = \mathbf{L}_1$ we can omit $\widetilde{\mathbf{L}}_0$ from this product. Likewise, since each $g \in \widetilde{\mathbf{L}}_n$ has $gv_S(\lambda_n) = cv_S(\lambda)$ for some $c \in \mathbb{C}^\times$ for a unique $\lambda \in \mathbf{L}_n \circledast \lambda_n$, we can omit $\mathbf{L}_n$ as well and rewrite our basis in the form given in the theorem. (3) then follows from dimensional considerations. □

Now that we have developed some basic properties of the characters χ_S, we can describe more elegantly how to construct them. In particular, we have the following result, which forms an analogue to Theorem 5.4 in [7]:

Theorem 7.3. Fix a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ of an algebra group $\mathbf{G}$, and let $\mathbf{H} \subseteq \mathbf{G}$ be the algebra subgroup given by

$$\mathbf{H} = \mathbf{LStab}_S(\lambda_n) = \{g \in \mathbf{L}_n \mid g \circledast_n \lambda_n = \lambda_n\}.$$

Define $\tau_S : \mathbf{H} \to \mathbb{C}^\times$ as the map given by by $\tau_S(g) = \alpha_{S,n}(g, \lambda_n)$ for $g \in \mathbf{H}$. The τ_S is a linear character of $\mathbf{H}$, and $\chi_S = (\tau_S)^{\mathbf{G}}$ is the character induced from τ_S.

Proof. Write $\tau = \tau_S$. That τ is a linear character of $\mathbf{H}$ is immediate, since by the Lemma 7.1 we have

$$ghv_S(\lambda_n) = \tau(h)gv_S(\lambda_n) = \tau(g)\tau(h)v_S(\lambda_n) = \tau(gh)v_S(\lambda_n)$$

for all $g, h \in \mathbf{H}$, so $\tau : \mathbf{H} \to \mathbb{C}^\times$ is a homomorphism. To prove the second half of the theorem, we observe that τ is the character of the $\mathbb{C}\mathbf{H}$-module $U = \mathbb{C}\mathbf{H}v_S(\lambda_n) = \mathbb{C}v_S(\lambda_n)$. It follows from (2) of the preceding theorem that we have a natural isomorphism $\mathrm{Ind}_{\mathbf{H}}^{\mathbf{G}}(U) = \mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{H}} U = \mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{H}} \mathbb{C}v_S(\lambda_n) \cong \mathbb{C}\mathbf{G}v_S(\lambda_n) = V_S$, so $\chi_S = \tau^{\mathbf{G}}$. □

This result gives rise to a natural corollary, stated below.

Corollary 7.1. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$, let W_S denote the left $\mathbb{C}\mathbf{L}_n$-module

$$W_S = \mathbb{C}\mathbf{L}_n v_S(\lambda_n) = \mathbb{C}\text{-span}\{v_S(\lambda) \mid \lambda \in \mathbf{L}_n \circledast \lambda_n\}$$

and let ω_S be its character. Then $\chi_S = (\omega_S)^{\mathbf{G}}$ is the character induced from ω_S.

Proof. It follows by the argument given in the proof of the preceding theorem that $\omega_S = (\tau_S)^{\mathbf{L}_n}$, so $\chi_S = \chi_S = (\omega_S)^{\mathbf{G}}$. □

We can give an explicit formula for the character ω_S which generalizes the supercharacter formula (4.3). In particular, we have the following result.

Theorem 7.4. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$,

$$\omega_S(g) = \frac{|\mathbf{L}_n \circledast \lambda_n|}{|\mathbf{L}_n \circledast \lambda_n \circledast \mathbf{R}_n|} \sum_{\lambda \in \mathbf{L}_n \circledast \lambda_n \circledast \mathbf{R}_n} \alpha_S(g, \lambda),$$

for all $g \in \mathbf{L}_n$.

Observe that if $n = 1$, then $\chi_S = \omega_S$ and $\mathbf{L}_1 = \mathbf{R}_1 = \mathbf{G}$, so this result precisely describes supercharacter formula given in Section 4. In order to prove the theorem, we require two lemmas.

Lemma 7.4. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. If $g \in \mathbf{L}_i$ and $\lambda \in \mathfrak{h}_i$ such that $g \circledast \lambda = \lambda$, then $\alpha_S(g, \lambda) = \alpha_S(g, \gamma)$ for all $\gamma \in \lambda \circledast \mathbf{R}_i$.

Proof. Clearly the lemma holds for $i = 0$, so suppose $i > 0$. Fix $g \in \mathbf{L}_i$ and $\lambda \in \mathfrak{h}_i$, and suppose there exists $\gamma = \lambda \circledast h$ for some $h \in \mathbf{R}_i$ such that $\alpha_S(g, \gamma) \neq \alpha_S(g, \lambda)$. We wish to show that this implies $g \circledast \lambda \neq \lambda$. One can check that $\alpha_S(g, \gamma) \neq \alpha_S(g, \lambda)$ implies $(\lambda \circledast h - \lambda)(g^{-1} \circledast \lambda_{i-1} - \lambda_{i-1}) \neq 0$. Since by Lemma 6.1 $(g \circledast \lambda - \lambda)(\lambda_{i-1} \circledast h^{-1} - \lambda_{i-1}) = (\lambda \circledast h - \lambda)(g^{-1} \circledast \lambda_{i-1} - \lambda_{i-1})$, it follows that $g \circledast \lambda \neq \lambda$, as desired. □

Lemma 7.5. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. If $g \in \mathbf{L}_i$ and $\lambda \in \mathfrak{h}_i$ such that $g \circledast \lambda \neq \lambda$, then $\sum_{\gamma \in \lambda \circledast \mathbf{R}_i} \alpha_S(g, \gamma) = 0$.

Proof. Fix $\lambda \in \mathfrak{h}_i$ and let $\mathfrak{r} = \{\lambda \circledast h - \lambda \mid h \in \mathbf{R}_i\}$ as in Lemma 6.2. Suppose $g \in \mathbf{L}_i$ such that $g \circledast \lambda \neq \lambda$, and note then that necessarily $i > 0$. Since $\mathfrak{r}$ is a subspace, the evaluation map $\xi : \mathfrak{r} \to \mathbb{C}^\times$ given by $\xi(\gamma) = \theta\left(\gamma(g^{-1} \circledast \lambda_{i-1} - \lambda_{i-1})\right)$ is a homomorphism. By assumption $g \circledast \lambda(X) \neq \lambda(X)$ for some $X = \lambda_{i-1} - \lambda_{i-1} \circledast h^{-1} \in \mathfrak{i}_{i-1}$ where $h \in \mathbf{R}_i$. If $Y = \lambda_{i-1} - g^{-1} \circledast \lambda_{i-1} \in \mathfrak{i}_{n-1}$, then by Lemma 6.1 we have $(g^{-1} \circledast \lambda - \lambda)(X) = (\lambda \circledast h - \lambda)(Y) \neq 0$ so $\xi(\lambda \circledast h - \lambda) \neq 1$. Thus ξ is nontrivial so $\sum_{\gamma \in \mathfrak{r}} \xi(\gamma) = 0$ and $\sum_{\gamma \in \lambda \circledast \mathbf{R}_i} \alpha_S(g, \gamma) = \alpha_S(g, \lambda) \sum_{\gamma \in \mathfrak{r}} \xi(\gamma) = 0$. □

We can now prove the theorem.

Proof of Theorem 7.4. By the preceding lemmas, if $g \in \mathbf{L}_n$ and $\lambda \in \mathfrak{h}_n$ then

$$gv_S(\lambda)\big|_{v_S(\lambda)} = \frac{1}{|\lambda \circledast \mathbf{R}_n|} \sum_{\gamma \in \lambda \circledast \mathbf{R}_n} \alpha_S(g, \gamma) = \begin{cases} \alpha_S(g, \lambda), & \text{if } g \circledast \lambda = \lambda \\ 0, & \text{otherwise.} \end{cases}$$

Let $\rho_1, \ldots, \rho_k \in \mathbf{L}_n \circledast \lambda_n$ be representatives of the distinct right orbits in $\mathbf{L}_n \circledast \lambda_n \circledast \mathbf{R}_n$. Then

$$\begin{aligned}\omega_S(g) &= \sum_{\lambda \in \mathbf{L}_n \circledast \lambda_n} \frac{1}{|\lambda_n \circledast \mathbf{R}_n|} \sum_{\gamma \in \lambda \circledast \mathbf{R}_n} \alpha_S(g,\gamma) = \sum_{i=1}^{k} \frac{|\mathbf{L}_n \circledast \lambda_n \cap \rho_k \circledast \mathbf{R}_n|}{|\lambda_n \circledast \mathbf{R}_n|} \sum_{\gamma \in \rho_k \circledast \mathbf{R}_n} \alpha_S(g,\gamma) \\ &= \sum_{i=1}^{k} \frac{|\mathbf{L}_n \circledast \lambda_n \cap \lambda_n \circledast \mathbf{R}_n|}{|\lambda_n \circledast \mathbf{R}_n|} \sum_{\gamma \in \rho_k \circledast \mathbf{R}_n} \alpha_S(g,\gamma) = \frac{|\mathbf{L}_n \circledast \lambda_n|}{|\mathbf{L}_n \circledast \lambda_n \circledast \mathbf{R}_n|} \sum_{i=1}^{k} \sum_{\gamma \in \rho_k \circledast \mathbf{R}_n} \alpha_S(g,\gamma) \\ &= \frac{|\mathbf{L}_n \circledast \lambda_n|}{|\mathbf{L}_n \circledast \lambda_n \circledast \mathbf{R}_n|} \sum_{\lambda \in \mathbf{L}_n \circledast \lambda_n \mathbf{R}_n} \alpha_S(g,\lambda).\end{aligned}$$

□

Our next result shows that, like supercharacters, the characters χ_S has disjoint constituents which partition $\mathrm{Irr}(\mathbf{G})$. Before proceeding, we recall some relevant notation. Given an arbitrary set of irreducible characters $\mathcal{X} \subseteq \mathrm{Irr}(\mathbf{G})$, let $\sigma_{\mathcal{X}}$ denote the character $\sigma_{\mathcal{X}} = \sum_{\psi \in \mathcal{X}} \psi(1)\psi$. Recall that each two-sided ideal $\mathcal{I}$ of a group algebra $\mathbb{C}\mathbf{G}$ is the direct sum of the minimal ideals that it contains. These minimal ideals, moreover, correspond to the irreducible characters of G, and if $\mathcal{X}$ is the set of irreducible characters corresponding to minimal ideals contained in the ideal $\mathcal{I}$, then $\mathcal{I}$, viewed as a left $\mathbb{C}\mathbf{G}$-module, affords the character $\sigma_{\mathcal{X}}$. We now have the following result, which generalizes Theorem 5.5 in [7]:

Theorem 7.5. Fix a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and let $\mathcal{X} \subseteq \mathrm{Irr}(\mathbf{G})$ denote the set of irreducible constituents of χ_S. Then the following hold:

(1) $m_S \chi_S = \sigma_{\mathcal{X}}$.

(2) Let $\mathscr{I} \subseteq \mathfrak{h}_S$ be a set of representatives of the two-sided orbits of $\mathfrak{h}_S$ under the $\circledast$-actions of $\mathbf{L}_S$ and $\mathbf{R}_S$. If $\mathcal{X}_\lambda$ is the set of irreducible constituents of $\chi_{S|\lambda}$ for $\lambda \in \mathscr{I}$, then the sets $\mathcal{X}_\lambda$ partition $\mathcal{X}$.

Proof. By definition M_S is a two-sided ideal in $\mathbb{C}\mathbf{G}$ given by a direct sum of a number of copies of V_S; in particular, given by the direct sum of all right translates of V_S. As such, (1) is equivalent to the statement that the $\mathbb{C}\mathbf{G}$-module M_S affords the character $m_S\chi_S$, or that M_S is a direct sum of m_S copies of V_S. We prove this by induction on the length n of the sequence S. If $n = 0$ then $v_S(\lambda_n) = 1 + \lambda_n \in \mathfrak{n}$ and $M_S = \mathbb{C}\mathbf{G} = V_S$, so clearly $M_S = V_S^{\oplus m_S}$ as $m_S = 1$. Suppose $n > 0$ and this still holds. Then for any $\lambda \in \mathfrak{h}_S$, it follows from Lemma 7.1 that each copy of V_S in M_S contains $\frac{m_{S|\lambda}}{m_S} = \frac{|\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S|}{|\mathbf{L}_S \circledast \lambda|}$ copies of $V_{S|\lambda}$. Since $M_{S|\lambda}$ consists of all right translates of $V_{S|\lambda}$ and since each of these modules must appear as a submodule of some right translate of V_S, it follows that $M_{S|\lambda} = V_{S|\lambda}^{\oplus m_{S|\lambda}}$. Hence $M_S = V_S^{\oplus m_S}$ for all decomposition sequences of length $n+1$, and so by induction the same is true for all n. Therefore $m_S\chi_S = \sigma_{\mathcal{X}}$.

To prove (2), we observe that since $\mathfrak{h}_S$ is a disjoint union of the two-sided orbits $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$ for $\lambda \in \mathscr{I}$, it follows from Lemma 7.1 that M_S is the direct of the submodules $M_{S|\lambda}$ for $\lambda \in \mathscr{I}$. Since M_S affords the characters $m_S\chi_S$, it follows that

$$\sigma_{\mathcal{X}} = m_S\chi_S = \sum_{\lambda \in \mathscr{I}} m_{S|\lambda}\chi_{S|\lambda} = \sum_{\lambda \in \mathscr{I}} \sigma_{\mathcal{X}_\lambda}.$$

By definition the multiplicity of each irreducible constituent of one of the characters $\sigma_{\mathcal{X}_\lambda}$ for $\lambda \in \mathscr{I}$ is equal to its degree, which is also equal to its multiplicity in $\sigma_{\mathcal{X}}$. Therefore the sets $\mathcal{X}_\lambda$ for $\lambda \in \mathscr{I}$ must be disjoint. Since their union is $\mathcal{X}$, the sets $\mathcal{X}_\lambda$ form a partition of $\mathcal{X}$. □

Recall the familiar inner product $\langle\ ,\ \rangle$ on the space of complex-valued functions $\mathbf{G} \to \mathbb{C}$ defined by

$$\langle \chi, \psi \rangle = \frac{1}{|\mathbf{G}|} \sum_{g \in \mathbf{G}} \chi(g)\overline{\psi(g)}$$

for $\chi, \psi : \mathbf{G} \to \mathbb{C}$. The following lemma tells us how to evaluate the inner product $\langle \chi_{S|\lambda}, \chi_{S|\gamma} \rangle$ given a decomposition sequence $S \in \mathscr{D}(\mathbf{G})$ and $\lambda, \gamma \in \mathfrak{h}_S$, and this gives us a condition for determining when the module V_S is irreducible.

Theorem 7.6. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and $\lambda, \gamma \in \mathfrak{h}_S$,

$$\langle \chi_{S|\lambda}, \chi_{S|\gamma} \rangle = \begin{cases} |\mathbf{L}_S \circledast \lambda \cap \lambda \circledast \mathbf{R}_S|, & \text{if } \gamma \in \mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Given an arbitrary subset $\mathcal{X} \subseteq \mathrm{Irr}(\mathbf{G})$, recall that $\sigma_{\mathcal{X}}(1) = \sum_{\psi \in \mathcal{X}} \psi(1)^2 = \langle \sigma_{\mathcal{X}}, \sigma_{\mathcal{X}} \rangle$. Since $m_{S|\lambda}\chi_{S|\lambda} = \sigma_{\mathcal{X}}$ for some $\mathcal{X} \subseteq \mathrm{Irr}(\mathbf{G})$, we have $m_{S|\lambda}^2 \langle \chi_{S|\lambda}, \chi_{S|\lambda} \rangle = \langle m_{S|\lambda}\chi_{S|\lambda}, m_{S|\lambda}\chi_{S|\lambda} \rangle = \langle \sigma_{\mathcal{X}}, \sigma_{\mathcal{X}} \rangle = \sigma_{\mathcal{X}}(1) = m_{S|\lambda}\chi_{S|\lambda}(1)$. Hence

$$\langle \chi_{S|\lambda}, \chi_{S|\lambda} \rangle = \frac{\chi_{S|\lambda}(1)}{m_{S|\lambda}} = \frac{|\mathbf{G}|}{|\mathbf{LStab}_{S|\lambda}(\lambda)|} \frac{|\mathbf{LStab}_{S|\lambda}(\lambda)|}{|\mathbf{G}|/|\mathfrak{i}_\lambda|} = |\mathfrak{i}_\lambda| = |\mathbf{L}_S \circledast \lambda \cap \lambda \circledast \mathbf{R}_S|.$$

The first half of the theorem now follows from the fact that $\chi_{S|\lambda} = \chi_{S|\gamma}$ if $\gamma \in \mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$. Alternatively, if $\gamma \notin \mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$ then $\chi_{S|\lambda}$ and $\chi_{S|\gamma}$ have disjoint sets of irreducible constituents, in which case $\langle \chi_{S|\lambda}, \chi_{S|\gamma} \rangle = 0$. □

Interpreting this result with different notation, we have the following corollary.

Corollary 7.2. If $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$, then $\langle \chi_S, \chi_S \rangle = |\mathfrak{i}_n|$. Hence the module V_S and character χ_S are irreducible if and only if $\mathfrak{i}_n = \{\mathbf{0}\}$.

We conclude this section with an analogue of Corollary 5.12 in [7].

Corollary 7.3. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ and suppose χ_S has a linear constituent. Then $\mathbf{L}_i \circledast \lambda_i = \mathbf{L}_i \circledast \lambda_i \circledast \mathbf{R}_i = \lambda_i \circledast \mathbf{R}_i$ for all $0 \leq i \leq n$.

Proof. Since $m_S\chi_S = \sigma_{\mathcal{X}} = \sum_{\psi \in \mathcal{X}} \psi(1)\psi$ for some $\mathcal{X} \subseteq \mathrm{Irr}(\mathbf{G})$, m_S must divide the degree of each irreducible constituent $\psi \in \mathcal{X}$. If one of these constituents is linear so that its degree is one, then we must have $m_S = 1$. But m_S is the product of the number of left orbits and the product of the number of right orbits in $\mathbf{L}_i \circledast \lambda_i \circledast \mathbf{R}_i$, so each $\mathbf{L}_i \circledast \lambda_i \circledast \mathbf{R}_i$ must be given by one left orbit and by one right orbit. □

We summarize the results of this section as follows. Given a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$, we defined a certain map $v_{S,i} : \mathfrak{h}_i \to \mathbb{C}\mathbf{G}$, and used this map to define a module V_S with character χ_S. We then showed how to decompose V_S into submodules by extending the sequence S, and described how the information stored in S determines a basis for V_S and its dimension. In addition, we described how to determine if V_S is irreducible, and proved that $m_S\chi_S = \sigma_{\mathcal{X}}$ for some subset $\mathcal{X} \subseteq \mathrm{Irr}(\mathbf{G})$. Finally, we showed that each χ_S is induced from a linear character of an algebra subgroup of $\mathbf{G}$. In the next section we discuss how to choose a set of representative sequences in $\mathscr{D}(S)$ to decompose the group algebra $\mathbb{C}\mathbf{G}$, and show that the resulting decomposition is uniquely determined by $\mathbf{G}$.

8 Decomposition trees

Constructing a decomposition sequence certainly involves a number of more or less arbitrary choices of representatives λ_i. In this section we show that such choices have no effect on the characters we ultimately obtain, and that any reasonable method of inductively building up a "representative" set of all decomposition sequences of an algebra group results in the same set of characters.

The results of the last section show that given $S \in \mathscr{D}(\mathbf{G})$, we can decompose the module V_S into a direct sum of non-isomorphic submodules by computing a set $\mathscr{I}$ of representatives of the two-sided $\circledast$-orbits in $\mathfrak{h}_S$ and then forming the descendent sequences $S|\lambda$ and corresponding modules $V_{S|\lambda}$ for $\lambda \in \mathscr{I}$. It follows that we can decompose the group algebra $\mathbb{C}\mathbf{G}$ into a direct sum of non-isomorphic submodules by beginning with the unique one term decomposition sequence $R = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n})\} \in \mathscr{D}(\mathbf{G})$, and then inductively applying the preceding process to V_R and all resulting submodules. In this way we can construct an infinite rooted subtree $\mathscr{T} \subseteq \mathscr{D}(\mathbf{G})$ of decomposition sequences with the following properties:

1. The root of $\mathscr{T}$ is $R = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n})\} \in \mathscr{D}(\mathbf{G})$.
2. If the children of $S \in \mathscr{T}$ are $\{S_i\}$, then $\chi_S = \frac{1}{m_S} \sum_i m_{S_i} \chi_{S_i}$ and $\langle \chi_{S_i}, \chi_{S_j} \rangle = 0$ if $i \neq j$.
3. If $\{S_i\} \subseteq \mathscr{T}$ are the nodes of $\mathscr{T}$ of height h, then the character of $\mathbb{C}\mathbf{G}$ is $\chi_{\mathbf{G}} = \sum_i m_{S_i} \chi_{S_i}$.

We call such a structure a *decomposition tree* of $\mathbf{G}$. To make this notion precise, we give the following definition:

Definition 5. A *decomposition tree* $\mathscr{T}$ of an algebra group $\mathbf{G}$ is an infinite subtree of $\mathscr{D}(\mathbf{G})$ with the following properties:

(1) $\mathscr{T}$ contains the unique one term decomposition sequence $R = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n})\} \in \mathscr{D}(\mathbf{G})$.

(2) If $S \in \mathscr{T}$, then there exists a set of representatives $\mathscr{I} \subseteq \mathfrak{h}_S$ of the two-sided $\circledast$-orbits in $\mathfrak{h}_S$ such that $S|\lambda \in \mathscr{T}$ for $\lambda \in \mathfrak{h}_S$ if and only if $\lambda \in \mathscr{I}$.

By construction, a decomposition tree $\mathscr{T}$ indexes a tree of characters of $\mathbf{G}$. To refer to this set, we have the following notation:

Notation. Given any subset $\mathscr{S} \subseteq \mathscr{D}(\mathbf{G})$ let $\hat{\mathscr{S}} = \{\chi_S \mid S \in \mathscr{S}\}$ denote the set formed by replacing each sequence in $\mathscr{S}$ with the character of $\mathbf{G}$ which it indexes.

Under this notation, for any decomposition tree $\mathscr{T}$, the set $\hat{\mathscr{T}}$ forms an infinite tree whose root is the character $\chi_{\mathbf{G}}$ of regular representation of $\mathbf{G}$. Each node of $\hat{\mathscr{T}}$ is a constituent of its parent which decomposes as as sum of its children. More strongly, by Lemma 7.5, the irreducible constituents of sibling nodes in $\hat{\mathscr{T}}$ partition the irreducible constituents of their parent.

The decomposition tree $\mathscr{T} \subseteq \mathscr{D}(\mathbf{G})$ clearly depends on our choices of representatives at each stage in its construction. The following theorem shows, however, that the character tree $\hat{\mathscr{T}}$ is independent of $\mathscr{T}$ and depends only on $\mathbf{G}$. Due to this result, we can index all of the modules V_S for $S \in \mathscr{D}(\mathbf{G})$ by constructing a single arbitrary decomposition tree $\mathscr{T}$.

Theorem 8.1. If $\mathscr{T}, \mathscr{T}' \subseteq \mathscr{D}(\mathbf{G})$ are decomposition trees of an algebra group $\mathbf{G} = 1 + \mathfrak{n}$, then $\mathscr{T} \cong \mathscr{T}'$ and $\hat{\mathscr{T}} = \hat{\mathscr{T}}'$. In other words, there exists a graph isomorphism $f : \mathscr{T} \to \mathscr{T}'$ such that $\chi_S = \chi_{f(S)}$ for all $S \in \mathscr{T}$.

To prove this theorem, we require the following lemma:

Lemma 8.1. Suppose $S, S' \in \mathscr{D}(\mathbf{G})$ are decomposition sequences such that $\chi_S = \chi_{S'}$. Then $\{\chi_{S|\lambda} \mid \lambda \in \mathfrak{h}_S\} = \{\chi_{S'|\lambda} \mid \lambda \in \mathfrak{h}_{S'}\}$.

Proof. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ and $S' = \{(\lambda'_i, \mathfrak{h}'_i, \mathbf{L}'_i, \mathbf{R}'_i, \mathfrak{i}'_i)\}_{i=0}^{n'}$ with $\chi_S = \chi_{S'}$. First suppose $n = n'$ and all terms of the two sequences agree except $\lambda_n \neq \lambda'_n$ and $\mathfrak{i}_n \neq \mathfrak{i}'_n$. Then $\lambda'_n = g \circledast \lambda_n \circledast h$ for some $g \in \mathbf{L}_n$ and $h \in \mathbf{R}_n$, and it follows directly that $\mathbf{L}_{S'} = g\mathbf{L}_S g^{-1}$, $\mathbf{R}_{S'} = h^{-1}\mathbf{R}_n h$, and $\mathfrak{i}'_n = g \circledast \mathfrak{i}_n \circledast h - \Phi$ where

$$\Phi(X) = X(g \circledast \lambda_{n-2} \circledast h - \lambda_{n-2}), \qquad \text{for } X \in \mathfrak{i}'_{n-1}.$$

Given these observations, it follows by an argument similar to the proof of Lemma 7.1 that for any $\lambda \in \mathfrak{h}_S$, $gv_{S|\lambda}(\lambda)h = kv_{S'|\lambda'}(\lambda')$ for some $k \in \mathbb{C}$ and $\lambda' \in \mathfrak{h}_S$. Therefore each $\chi_{S|\lambda}$ is equal to some $\chi_{S'|\lambda'}$, so $\{\chi_{S|\lambda} \mid \lambda \in \mathfrak{h}_S\} \subseteq \{\chi_{S'|\lambda} \mid \lambda \in \mathfrak{h}_{S'}\}$, and the reverse containment follows by symmetry.

Suppose $n = n'$ but S and S' differ in more than the last term. If the two sequences agree in the first k terms but $\lambda_k \neq \lambda'_k$, then we must have $\lambda'_k = g \circledast \lambda_k \circledast h$ for some $g \in \mathbf{L}_k$

and $h \in \mathbf{R}_k$. One can then show by induction that set the modules of the form $gV_{S|\lambda}h$ are equal to the set of modules of the form $V_{S''|\lambda''}$ for some decomposition sequence S'' of length n which agrees with S' in the first $k+1$ terms. Using this fact and inductively applying the preceding argument then shows that if $S, S' \in \mathscr{D}(\mathbf{G})$ has the same length and $\chi_S = \chi_{S'}$, then the two characters have the same children.

If $n \neq n'$, then without loss of generality we can assume that $n > n'$ and $\mathfrak{i}_{n-1} = \mathfrak{h}_{n-1}$. Let $R \in \mathscr{D}(\mathbf{G})$ be the subsequence of S given dropping the last term, and note that $\chi_R = \chi_S$ since the χ_S is a constituent of χ_R with the same degree. Now observe that since $\mathbf{L}_S = \mathbf{L}_n$ and $\mathbf{R}_S = \mathbf{R}_n$, it follows that $\mathbf{L}_S \circledast \mathbf{0} = \mathbf{0} \circledast \mathbf{R}_S = \mathfrak{h}_S$ for $\mathbf{0} \in \mathfrak{h}_S$. Hence there is only one left, right, and two-sided $\circledast$-orbit in $\mathfrak{h}_S$, all of which coincide, and so $\{\chi_{S|\lambda} \mid \lambda \in \mathfrak{h}_S\} = \{\chi_{R|\lambda} \mid \lambda \in \mathfrak{h}_R\} = \{\chi_S\}$. Therefore we can replace S with R, and by repeatedly applying such substitutions we can assume $n = n'$ and turn to the preceding argument. $\square$

We can now prove the theorem.

Proof of Theorem 8.1. Let the root nodes of $\mathscr{T}$ and $\mathscr{T}'$ be the unique one-term decomposition sequence $R \in \mathscr{D}(\mathbf{G})$. Define $f : \mathscr{T} \to \mathscr{T}'$ as the unique map with the following properties:

(1) $f(R) = R$.

(2) If $S \in \mathscr{T}$ and $S|\lambda \in \mathscr{T}$ for some $\lambda \in \mathfrak{h}_S$, then $f(S|\lambda) = S'|\lambda'$ where $S' = f(S)$ and $\lambda' \in \mathfrak{h}_{S'}$ is the unique representative such that $S'|\lambda' \in \mathscr{T}'$ and $\chi_{S|\lambda} = \chi_{S'|\lambda'}$.

By the lemma, this map is a well-defined graph isomorphism, and by construction $\chi_S = \chi_{f(S)}$ for all $S \in \mathscr{T}$. $\square$

Lemma 8.1 yields the following corollary, which shows that in addition to being unique, the character tree $\hat{\mathscr{T}}$ is effectively finite:

Corollary 8.1. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$ be a decomposition sequence. If $n > 0$ and $\mathfrak{h}_n = \mathfrak{i}_n$, then $\chi_S = \chi_T$ for every $T \in \mathscr{D}(\mathbf{G})$ which is a descendent of S.

Proof. If $\mathfrak{h}_n = \mathfrak{i}_n$ then $\mathbf{L}_n \circledast \lambda_n = |\mathfrak{h}_n|$. Assume $n > 0$ and let $S' = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^{n-1}$. Then it follows from Theorem 7.2 that $\chi_S(1) = \chi_{S'}(1)$, so since χ_S is a constituent of $\chi_{S'}$, we must have $\chi_S = \chi_{S'}$. Therefore by the lemma we have $\{\chi_{S|\lambda} \mid \lambda \in \mathfrak{h}_S\} = \{\chi_{S'|\lambda} \mid \lambda \in \mathfrak{h}_{S'}\}$, and since there is evidently only one two-sided $\circledast$-orbit in $\mathfrak{h}_n$, it follows that $\{\chi_{S|\lambda} \mid \lambda \in \mathfrak{h}_S\} = \{\chi_{S'|\lambda} \mid \lambda \in \mathfrak{h}_{S'}\} = \{\chi_S\}$. Therefore by induction $\chi_S = \chi_T$ for every descendent T of S. $\square$

Recall that the *rank* of a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n \in \mathscr{D}(\mathbf{G})$, denoted rank$(S)$, is the least positive integer $r \leq n$ such that $|\mathfrak{i}_r| = |\mathfrak{i}_{r+1}|$, or n if no such r exists. We say that a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ is *full rank* if rank$(S) = n$. The preceding corollary then shows that every character χ_S for $S \in \mathscr{D}(\mathbf{G})$ is obtained from a decomposition sequence with full rank. To index such characters, we define a *complete decomposition tree* of an algebra group $\mathbf{G}$ as the subtree of full rank sequences in some decomposition tree. A complete decomposition tree is necessarily finite, and indexes all the distinct characters χ_S for $S \in \mathscr{D}(\mathbf{G})$. Intuitively, one forms a complete decomposition tree by cutting off the infinite branches of degenerate sequences in an ordinary decomposition tree.

Given these definitions, we can now summarize the main results of this section with the following theorem:

Theorem 8.2. Let $\mathscr{T} \subseteq \mathscr{D}(\mathbf{G})$ be a complete decomposition tree of an algebra group $\mathbf{G} = 1 + \mathfrak{n}$, and let $\mathscr{L} \subseteq \mathscr{T}$ denote the set of its leaf nodes. Then the following hold:

(1) The character tree $\hat{\mathscr{T}} = \{\chi_S \mid S \in \mathscr{T}\}$ is independent of $\mathscr{T}$ and depends only on $\mathbf{G}$, and is equal to $\{\chi_S \mid S \in \mathscr{D}(\mathbf{G})\}$ as a set.

(2) If $S \in \mathscr{T}$ and $\mathscr{L}_S \subseteq \mathscr{L}$ is the set of leaf nodes which are descendants of S, then the character χ_S decomposes as the sum

$$\chi_S = \frac{1}{m_S} \sum_{T \in \mathscr{L}_S} m_T \chi_T.$$

In particular, the character $\chi_{\mathbf{G}}$ of the regular representation of $\mathbf{G}$ decomposes as the sum

$$\chi_{\mathbf{G}} = \sum_{S \in \mathscr{L}} m_S \chi_S.$$

(3) If $S, T \in \mathscr{T}$, then $\langle \chi_S, \chi_T \rangle = 0$ unless T is a descendent of S or vice versa. Thus every irreducible character of $\mathbf{G}$ appears as a constituent of χ_S for exactly one $S \in \mathscr{L}$.

By Corollary 8.1 the recursive constructions described in the preceding sections fail to decompose the leaf characters χ_S for $S \in \mathscr{L}$ any further. One might hope that, as result, all the characters in $\hat{\mathscr{L}}$ are irreducible, but in general this does not hold, at least over $\mathbb{C}$. For a simple counterexample, consider the following:

Example. Let $\mathbf{G} = 1 + \mathfrak{n}$ be the algebra group over $\mathbb{F}_2$ given by

$$\mathbf{G} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{F}_2 \right\} = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}_4.$$

$\mathbf{G}$ is isomorphic to the cyclic group of order four, and so has two irreducible characters with non-real values. Since the image of the homomorphism $\theta : \mathbb{F}_2^+ \to \mathbb{C}^\times$ is just $\{-1, 1\}$, our constructions only generate real characters so the leaf nodes of the character tree $\hat{\mathscr{T}}$ cannot possibly be all irreducible. To see this explicitly, note that we have $\mathfrak{n} = \mathbb{F}_2\text{-span}\{X, Y\}$ where

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $\mathfrak{n}^* = \mathbb{F}_2\text{-span}\{\eta, \rho\}$ where

$$\begin{matrix} \eta(X) = 1, \\ \eta(Y) = 0, \end{matrix} \qquad \text{and} \qquad \begin{matrix} \rho(X) = 0, \\ \rho(Y) = 1. \end{matrix}$$

Under this notation, $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^1 = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n}), (\eta, \mathfrak{n}^*, \mathbf{G}, \mathbf{G}, \mathfrak{n}^*)\}$ forms a two-term decomposition sequence with $\langle \chi_S, \chi_S \rangle = |\mathbf{L}_1 \circledast_1 \lambda_1 \cap \lambda_1 \circledast_1 \mathbf{R}_1| = |\{\eta, \eta + \rho\}| = 2$ and $\mathfrak{h}_1 = \mathfrak{i}_1 = \mathfrak{n}^*$. Hence χ_S is reducible, but by Corollary 8.1 our methods fail to decompose χ_S any further. One can check that χ_S is equal to the sum of the two non-real irreducible characters of $\mathbf{G}$, and that the other two irreducible characters of $\mathbf{G}$ are obtained from the two-term decompositions sequences with $\lambda_1 = 0$ and $\lambda_1 = \rho$.

We call sequences of this kind *degenerate*. That is, a decomposition sequence $S \in \mathscr{D}(\mathbf{G})$ is *degenerate* if χ_S is reducible but equal to all of its descendants. By Corollary 8.1, this is equivalent to saying that $|\mathfrak{i}_{n-1}| = |\mathfrak{i}_n| > 1$. Given this definition and the preceding counterexample, one naturally asks if characters indexed by degenerate sequences are in any sense irreducible. Answering this question goes beyond the scope of this work, but a reasonable conjecture might be that such characters are irreducible over $\mathbb{Q}_p$, the cyclotomic subfield of the complex numbers given by adjoining a primitive pth root of unity to the rationals. Observe that the image of our homomorphism $\theta : \mathbb{F}_q^+ \to \mathbb{C}^\times$ lies in $\mathbb{Q}_p$, and so all of our constructions thus far could be defined solely over this field. Admittedly we have little evidence at this time to support such a broad conjecture, but we also lack any counterexamples.

9 Decomposing $\mathbb{C}\mathbf{U}_n(\mathbb{F}_q)$

In this final section, we discuss how to apply the theory developed in this work to a particular family of algebra groups: namely, the groups $\mathbf{U}_n(\mathbb{F}_q)$ of $n \times n$ upper triangular matrices over $\mathbb{F}_q$ with ones on the diagonal. We have several reasons for examining this specific family. On the one hand, every algebra group appears as a subgroup of $\mathbf{U}_n(\mathbb{F}_q)$ for some choice of n and q, and so this family represents a natural starting point for nontrivial applications of our theory. At the same time, we can compute the decomposition sequences of $\mathbf{U}_n(\mathbb{F}_q)$ much more efficiently than those of a general algebra group due to several properties particular to $\mathbf{U}_n(\mathbb{F}_q)$. Such optimizations allow us to make computations which were formerly unfeasible. Finally, and most importantly, by explicitly computing the modules described in previous sections, we will be able to provide constructive proofs of some surprising properties of the irreducible characters of $\mathbf{U}_n(\mathbb{F}_q)$.

In particular, [11] proves that the irreducible characters of $\mathbf{U}_n(\mathbb{F}_2)$ are real-valued for $n < 13$ but that, remarkably, $\mathbf{U}_{13}(\mathbb{F}_2)$ has exactly two irreducible characters with non-real values. The proof of this result comes from a recursive formula for counting the number of involutions of $\mathbf{U}_n(\mathbb{F}_2)$, and as such is highly non-constructive. We will be able to describe the modules of these characters explicitly in terms of the theory developed in preceding sections. Using these constructions we will be able to answer several formerly inaccessible questions regarding the basic properties of these characters.

In this direction, we first observe that for any algebra group $\mathbf{G} = 1 + \mathfrak{n}$, we have the following general algorithm for computing a complete decomposition tree $\mathscr{T}$:

Algorithm 1. Let $\mathbf{G} = 1 + \mathfrak{n}$. To compute a complete decomposition tree $\mathscr{T}$ of $\mathbf{G}$:

1. Declare a queue of decomposition sequences `Q` and an empty decomposition tree $\mathscr{T}$.
2. Add the one term decomposition sequence $R = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n})\}$ to `Q`.
3. While `Q` is not empty, repeat:
   a. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ be the next element dequeued from `Q`.
   b. If $|\mathfrak{i}_n| > 1$ and $|\mathfrak{i}_n| \neq |\mathfrak{h}_n|$:
      i. Compute coset representatives of $\mathbf{L}_S/\mathbf{LStab}_S(\lambda_{n-1})$ and $\mathbf{R}_S/\mathbf{RStab}_S(\lambda_{n-1})$.
      ii. Compute the two-sided orbits $\circledast$ in $\mathfrak{h}_S$ and choose a set of representatives $\mathscr{I}$.
      iii. For each $\lambda \in \mathscr{I}$, compute $\mathfrak{i}_\lambda$.
      iv. Enqueue the decomposition sequences $S|\lambda$ for $\lambda \in \mathscr{I}$ in `Q`.
   c. Insert S into the tree $\mathscr{T}$.
4. Return $\mathscr{T}$.

In order to implement this algorithm efficiently, one should make use of the following basic optimizations. First, because for any decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ we have $\mathbf{LStab}_S(\lambda_{i-1}) \subseteq \mathbf{LStab}_S(\lambda_i) \subseteq \mathbf{L}_i$ and $\mathbf{RStab}_S(\lambda_{i-1}) \subseteq \mathbf{RStab}_S(\lambda_i) \subseteq \mathbf{R}_i$ for all i, it is not necessary to store or compute the groups $\mathbf{L}_i$ and $\mathbf{R}_i$ at each stage. Instead, it suffices on each iteration to find a set of coset representatives of the quotient groups $\mathbf{L}_i/\mathbf{LStab}_S(\lambda_{i-1})$ and $\mathbf{R}_i/\mathbf{LStab}_S(\lambda_{i-1})$. On the next iteration, one can then form a new set of representatives by taking a subset of the previous set. This optimization dramatically decreases the amount of time and memory required to partition $\mathfrak{h}_S$ into two-sided orbits on successive iterations.

Our second optimization concerns the computation of the $\circledast$-orbits of a given $\lambda \in \mathfrak{h}_S$. A natural way of computing all two-sided $\circledast$-orbits in $\mathfrak{h}_S$ goes as follows: form a set `h_S` containing all the elements of $\mathfrak{h}_S$, then repeatedly choose an arbitrary $\lambda \in$ `h_S`, compute its two-sided orbit, and remove those elements from `h_S` until the set is empty. Given an arbitrary $\lambda \in \mathfrak{h}_S$, one can efficiently compute $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$ by making use of the following facts given in Section 6:

(1) If $g \circledast \lambda = \lambda + \gamma$ for some $g \in \mathbf{L}_S$, then $g \circledast \lambda \circledast h = \lambda \circledast h + \gamma * h$ for any $h \in \mathbf{R}_S$.

(2) The $*$-action of $\mathbf{R}_S$ on $\mathfrak{h}_S$ is linear.

(3) The left orbit $\mathbf{L}_S \circledast \lambda$ is given by the affine set $\mathbf{L}_S \circledast \lambda = \lambda + \mathfrak{l}$, where $\mathfrak{l}$ is a subspace.

These observations lead to the following algorithm for simultaneously computing $\mathbf{L}_S \circledast \lambda$, $\lambda \circledast \mathbf{R}_S$, and $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$:

Algorithm 2. Let $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^n$ be a decomposition sequence and fix $\lambda \in \mathfrak{h}_S$. To compute the orbits $\mathbf{L}_S \circledast \lambda$, $\lambda \circledast \mathbf{R}_S$, and $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$:

1. Compute the left orbit $\mathbf{L}_S \circledast \lambda$ by evaluating $g \circledast \lambda$ for $g \in \mathbf{L}_S/\mathbf{LStab}_S(\lambda_n)$.
2. Find a basis $\mathcal{B}$ for the vector space $\mathfrak{l} = \mathbf{L}_S \circledast \lambda - \lambda$.
3. For each $h \in \mathbf{R}_S/\mathbf{RStab}_S(\lambda_n)$:
   a. Compute $\lambda \circledast h$ and $\gamma * h$ for $\gamma \in \mathcal{B}$.
   b. Add $\lambda \circledast h$ to the right orbit $\lambda \circledast \mathbf{R}_S$.
   c. Add $\lambda \circledast h + \mathbb{F}_q\text{-span}\{\gamma * h \mid \gamma \in \mathcal{B}\}$ to the two-sided orbit $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$.
4. Return $\mathbf{L}_S \circledast \lambda$, $\mathbf{R}_S \circledast \lambda$, and $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$.

Finding a basis $\mathcal{B}$ for $\mathfrak{l}$ and computing the vector space $\mathbb{F}_q\text{-span}\{\gamma * h \mid \gamma \in \mathcal{B}\}$ can generally be done much more efficiently than directly computing the right translates of $\mathbf{L}_S \circledast \lambda$. As a result, this algorithm typically represents a vast improve over naive methods for finding $\mathbf{L}_S \circledast \lambda \circledast \mathbf{R}_S$, since it effectively only requires the computation of the left and right orbits of $\lambda \in \mathfrak{h}_S$, which are already needed to compute $\mathfrak{i}_\lambda$.

These improvements apply to any algebra group, but several properties specific to the groups $\mathbf{U}_n(\mathbb{F}_q)$ allow us to speed up these algorithms even further. First, we can describe the rank one decomposition sequences of $\mathbf{U}_n(\mathbb{F}_q)$ explicitly in terms of familiar combinatorial objects for all n and q, and this allows us to complete the first iteration of Algorithm 1 is essentially constant time. The rank one decomposition sequences index the supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$, and from Section 5 it follows that every such sequence is of the form $S = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n}), (\lambda, \mathfrak{n}^*, \mathbf{G}, \mathbf{G}, \mathfrak{i}_\lambda)\}$ where $\mathbf{G} = \mathbf{U}_n(\mathbb{F}_q)$ and $\mathfrak{n} = \mathfrak{u}_n(\mathbb{F}_q)$, and $\lambda \in \mathfrak{n}^*$ is a linear functional represented by an $n \times n$ upper triangular matrix with zeros on the diagonal and with at most one nonzero position in each row and column. Given such an S, we can easily describe a basis for the vector space $\mathfrak{i}_\lambda$ and generating sets for the quotient groups $\mathbf{L}_S/\mathbf{LStab}_S(\lambda)$ and $\mathbf{R}_S/\mathbf{RStab}_S(\lambda)$. Our general algorithms provide ways of computing the latter groups but not their generators, and knowing these generators allows us to replace the iterative computation of the left and right orbits in Algorithm 2 with a more efficient recursive method. The following proposition states our result concerning these computations:

Notation. Let e_{ij} denote an $n \times n$ matrix with 1 in position (i, j) and zeros elsewhere.

Proposition 9.1. Let $\mathbf{G} = \mathbf{U}_n(\mathbb{F}_q)$ and $\mathfrak{n} = \mathfrak{u}_n(\mathbb{F}_q)$. Suppose $\lambda \in \mathfrak{n}^*$ is a linear functional viewed as an $n \times n$ upper triangular matrix with zeros on the diagonal and at most one nonzero position in each row and column. As usual, let $\text{supp}(\lambda)$ denote the set of positions (i, j) with $\lambda_{ij} \neq 0$. Now define three sets of positions by

$$\begin{aligned}
\mathcal{I}_\lambda &= \{(j,k) \mid \text{there are } i<j<k<l \text{ with } (i,k),(j,l) \in \text{supp}(\lambda)\},\\
\mathcal{L}_\lambda &= \{(i,j) \mid \text{there are } i<j<k<l \text{ with } (i,k),(j,l) \in \text{supp}(\lambda)\},\\
\mathcal{R}_\lambda &= \{(k,l) \mid \text{there are } i<j<k<l \text{ with } (i,k),(j,l) \in \text{supp}(\lambda)\}.
\end{aligned}$$

Let $S = \{(0, \mathfrak{n}, \mathbf{G}, \mathbf{G}, \mathfrak{n}), (\lambda, \mathfrak{n}^*, \mathbf{G}, \mathbf{G}, \mathfrak{i}_\lambda)\}$ be the rank one decomposition sequence indexed by λ. Then

$$\begin{aligned}
\mathfrak{i}_\lambda &= \mathbb{F}_q\text{-span}\{e_{jk} \in \mathfrak{n}^* \mid (j,k) \in \mathcal{I}_\lambda\},\\
\mathbf{G}/\mathbf{LStab}_S(\lambda) &= \langle 1 + te_{ij} \mid (i,j) \in \mathcal{L}_\lambda,\ t \in \mathbb{F}_q\rangle,\\
\mathbf{G}/\mathbf{RStab}_S(\lambda) &= \langle 1 + te_{kl} \mid (k,l) \in \mathcal{R}_\lambda,\ t \in \mathbb{F}_q\rangle,
\end{aligned}$$

and we can identify

$$\mathfrak{h}_S = \mathfrak{i}_\lambda^* \cong \mathbb{F}_q\text{-span}\{e_{jk} \in \mathfrak{n} \mid (j,k) \in \mathcal{I}_\lambda\},$$

where $\gamma \in \mathfrak{h}_S$ is evaluated at $X \in \mathfrak{i}_\lambda$ by $\gamma(X) = \sum_{(i,j)\in\mathcal{I}_\lambda} \gamma_{ij} X_{ij}$.

Proof. The left $\circledast$-action of $1 - te_{ij} \in \mathbf{G}$ on $\lambda \in \mathfrak{n}^*$ adds the ith row of λ multiplied by t to the jth row. Likewise, the right $\circledast$-action of $1 - te_{ij}$ on λ add the jth column of λ multiplied by t to the ith column. From this observation, it follows directly that every element in $\mathbf{G} \circledast \lambda$ is of the form $\lambda + \gamma$ where the support of γ is a subset of the positions above the diagonal which lie below positions in $\mathrm{supp}(\lambda)$. Likewise every element in $\lambda \circledast \mathbf{G}$ is of the form $\lambda + \gamma$ where $\mathrm{supp}(\gamma)$ is a subset of the positions which lie to the left of positions in $\mathrm{supp}(\lambda)$. Therefore the given basis for $\mathfrak{i}_\lambda$ follows by definition. To prove the other parts of the proposition, observe that if $(i,j) \in \mathcal{L}_\lambda$ has $i < j < k < l$ with $(i,k),(j,l) \in \mathrm{supp}(\lambda)$, then $(1 - te_{ij}) \circledast \lambda - \lambda = t\lambda_{ik} e_{jk} \in \mathfrak{i}_\lambda$ for any $t \in \mathbb{F}_q$. It follows with a little work that every element of $\mathfrak{i}_\lambda$ is of the form $g \circledast \lambda - \lambda$ where g is given by an appropriately ordered product of elements $1 - t_{ij}e_{ij}$ for $(i,j) \in \mathcal{L}_\lambda$ and $t_{ij} \in \mathbb{F}_q$. This results in the given generating set of $\mathbf{G}/\mathbf{LStab}_S(\lambda)$, and the argument for $\mathbf{G}/\mathbf{RStab}_S(\lambda)$ is identical. $\square$

The following example provides a visual explanation of the definitions given in this proposition.

Example. Suppose $\lambda \in \mathfrak{n}^*$ is given by

$$\lambda = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}.$$

The sets $\mathcal{I}_\lambda$, $\mathcal{L}_\lambda$, $\mathcal{R}_\lambda$ are then given by the positions marked $\blacksquare$ in the following matrices:

$$\underset{\mathcal{I}_\lambda}{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & \blacksquare & 0 & 0 & 0 \\ & & 0 & \blacksquare & \blacksquare & 0 & 0 \\ & & & 0 & \blacksquare & \blacksquare & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}}, \quad \underset{\mathcal{L}_\lambda}{\begin{pmatrix} 0 & \blacksquare & \blacksquare & 0 & 0 & 0 & 0 \\ & 0 & \blacksquare & \blacksquare & 0 & 0 & 0 \\ & & 0 & \blacksquare & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}}, \quad \underset{\mathcal{R}_\lambda}{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & \blacksquare & \blacksquare & 0 \\ & & & & 0 & \blacksquare & \blacksquare \\ & & & & & 0 & \blacksquare \\ & & & & & & 0 \end{pmatrix}}.$$

Using the algorithms described above with the accompanying optimizations, we were able to compute complete decompositions trees for $\mathbf{U}_n(\mathbb{F}_q)$ for $n \leq 12$. For $n \leq 10$, we made this calculation using a single computer with code written in Java. For each $n \leq 7$ the computation finished within a few seconds; for $n = 8$, it took around thirty seconds; for $n = 9$, it took a couple of minutes; and for $n = 10$, it took about half an hour. For $n = 11$ and $n = 12$ we divided the computations across about forty computers running remote processes on the Stanford network. With this increased computing power, we were able to calculate a complete decomposition tree for $\mathbf{U}_{11}(\mathbb{F}_2)$ within two hours and for $\mathbf{U}_{12}(\mathbb{F}_2)$ within eight hours. None of the resulting trees contained any degenerate sequences, and so we have the following theorem, which was shown non-constrively in [11]. To state it, recall that a character is *realizable* over a field $\mathbb{K}$ if χ is the character of a representation $\rho : \mathbf{G} \to \mathrm{GL}(V)$ whose image lies in a matrix ring over $\mathbb{K}$. If χ is realizable over $\mathbb{K}$ then its values lie in $\mathbb{K}$, but the converse is not true in general.

Theorem 9.1. If $n \leq 12$, then every character of $\mathbf{U}_n(\mathbb{F}_2)$ is realizable over $\mathbb{Q}$. In fact, every irreducible character of $\mathbf{U}_n(\mathbb{F}_2)$ is given by χ_S for some decomposition sequence $S \in \mathscr{D}(\mathbf{U}_n(\mathbb{F}_2))$.

A table of the number of irreducible characters obtained in each rank of our decomposition trees appears below. The totals in the last column are taken from the formulas given in [4], but one can check that these numbers are also given by the sum of the preceding columns. Observe that the Rank 1 column gives the familiar Catalan numbers, as these count the

n	Rank 0	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5	Rank 6	$\lvert\mathrm{Irr}(U_n(\mathbb{F}_2))\rvert$
1	1							1
2	0	2						2
3	0	5						5
4	0	14	2					16
5	0	42	19					61
6	0	132	141	2				275
7	0	429	974	27				1430
8	0	1430	6747	327	2			8506
9	0	4862	48594	3714	35			57205
10	0	16796	371881	42871	563	2		432113
11	0	58786	3062536	511573	8350	43		3641288
12	0	208012	27315986	6415025	124984	863	2	27523998

Figure 1: Numbers of irreducible characters of $U_n(\mathbb{F}_2)$ obtained in each rank

number of irreducible supercharacters of $\mathbf{U}_n(\mathbb{F}_2)$. The other columns do not appear to correspond to any well-known sequences.

For even n, there appear to be exactly two irreducible characters of $\mathbf{U}_n(\mathbb{F}_2)$ obtained in rank $n/2$. We can describe a simple construction for these characters over any finite field $\mathbb{F}_q$. Assume n is even, and consider the set partition λ given by $1, 3, 5, \ldots, n-1 \mid 2, 4, 6, \ldots, n$. In other words, let $\lambda \in \mathfrak{u}_n^*(\mathbb{F}_q)$ be the matrix with ones on the diagonal containing positions $(1, 3)$ and $(n-2, n)$:

$$\lambda = \begin{pmatrix} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}.$$

If $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^{n/2}$ is any decomposition sequence with $\lambda_2 = \lambda$, then $\mathfrak{h}_i$ has a single two-sided $\circledast$-orbit for $i < n/2 - 1$ while $\mathfrak{h}_{n/2-1}$ has exactly q orbits, each of size one. It follows that χ^λ has q irreducible constituents of rank $n/2$, so in particular the height of any complete decomposition tree of $\mathbf{U}_n(\mathbb{F}_q)$ is unbounded as a function of n.

We know that Theorem 9.1 does not hold for $n \geq 13$. In particular, [11] proves that $\mathbf{U}_{13}(\mathbb{F}_2)$ has exactly two irreducible characters with non-real values. As our final result, we provide an explicit construction of these characters using decomposition sequences. In particular, we demonstrate a degenerate decomposition sequence indexing a character with two irreducible constituents. We then show how to decompose this character into a sum $\chi + \overline{\chi}$, and explicitly evaluate χ on an element of $\mathbf{U}_{13}(\mathbb{F}_2)$ to prove that χ and $\overline{\chi}$ take non-real values.

The task of finding the degenerate sequence in question poses a non-trivial computational problem in its right. Computing a complete decomposition tree for $\mathbf{U}_{13}(\mathbb{F}_2)$ is a feasible but time consuming solution, given the growth rates displayed in this calculation for $n \leq 12$. Luckily, a simple observation vastly diminishes the number of sequences we need to consider in our search.

Consider the antitranspose map $\dagger$ which flips a matrix about the lower-left/upper-right diagonal. This map gives an outer automorphism of $\mathbf{U}_n(\mathbb{F}_q)$ and permutes the set of matrices indexing the supercharacters of $\mathbf{U}_n(\mathbb{F}_q)$—i.e., the upper triangular matrices given by diagrams of set partitions. It is easy to see from the supercharacter formula (4.3) that $\chi^\lambda \circ \dagger = \chi^{\dagger(\lambda)}$ for all $\lambda \in \mathfrak{u}_n^*(\mathbb{F}_q)$. Also, note that the two non-real irreducible characters $\chi, \overline{\chi}$ of $\mathbf{U}_{13}(\mathbb{F}_2)$ must appear as constituents of the same supercharacter since the supercharacters of $\mathbf{U}_n(\mathbb{F}_2)$ are always real-valued. Since $\chi \circ \dagger$ is also an irreducible character with non-real values, it follows that χ is a constituent of the supercharacter χ^λ only if χ is also a constituent of $\chi^\lambda \circ \dagger = \chi^{\dagger(\lambda)}$. As the constituents of distinct supercharacters are disjoint, we conclude the supercharacter χ^λ of $\mathbf{U}_{13}(\mathbb{F}_2)$ has a non-real valued constituent only if $\lambda = \dagger(\lambda)$.

Thus to find the character in question, we need only construct the constituents of the supercharacters χ^λ of $\mathbf{U}_{13}(\mathbb{F}_2)$ indexed by set partitions which are fixed points of the antitranspose map. The number of such set partitions in given by sequence A080107 in [13], and its thirteenth term is only 16033. This is less than the number of supercharacters of $\mathbf{U}_{10}(\mathbb{F}_2)$—which is 21147, the tenth Bell number—and so our previously formidable search has been reduced to an "almost" trivial computation. Of course the vector spaces $\mathfrak{i}_\lambda$ for set partitions of 13 are generally several orders of magnitude larger than the corresponding spaces for set partitions of 10, but this still does not render our search infeasible.

Using the same code as in our previous computations, we calculated all descendants of the supercharacters χ^λ with $\dagger(\lambda) = \lambda$. We found exactly one degenerate decomposition sequence, and by the preceding argument the two irreducible characters with non-real values must be constituents of the character indexed by this sequence. The following theorem states some consequences of this discovery, and its proof describes the construction of the corresponding degenerate sequence.

Theorem 9.2. The two complex irreducible characters $\chi, \overline{\chi}$ of $\mathbf{U}_{13}(\mathbb{F}_2)$ are induced from linear characters of the algebra group

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & 1 & a & \bullet & a & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & 1 & b & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & 1 & 0 & 0 & 0 & c & \bullet & 0 & \bullet & \bullet & \bullet \\ & & & & 1 & b & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & 1 & b & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & & 1 & c & \bullet & \bullet & \bullet & \bullet & \bullet \\ & & & & & & & 1 & \bullet & 0 & 0 & \bullet & \bullet \\ & & & & & & & & 1 & 0 & 0 & 0 & \bullet \\ & & & & & & & & & 1 & \bullet & \bullet & \bullet \\ & & & & & & & & & & 1 & \bullet & \bullet \\ & & & & & & & & & & & 1 & \bullet \\ & & & & & & & & & & & & 1 \end{pmatrix} \;\middle|\; a, b, c, \bullet \in \mathbb{F}_2 \right\},$$

where, as usual, we use the symbol $\bullet$ to label positions whose values in an element of $\mathbf{H}$ can be chosen independently of all other positions. In addition, the following hold:

(1) $\chi(1) = \overline{\chi}(1) = 2^{16}$, and all real-valued characters of $\mathbf{U}_{13}(\mathbb{F}_2)$ are realizable over $\mathbb{R}$.

(2) χ and $\overline{\chi}$ are constituents of the supercharacter indexed by the set partition

$$\lambda = 1,5,7,9,13 \;\big|\; 2,6,8,12 \;\big|\; 3,10 \;\big|\; 4,11$$

or equivalently by the 13×13 matrix

$$\lambda = e_{1,5} + e_{2,6} + e_{3,10} + e_{4,11} + e_{5,7} + e_{6,8} + e_{7,9} + e_{8,12} + e_{9,13}.$$

(3) χ and $\overline{\chi}$ are not the only constituents of this supercharacter. In particular, χ^λ decomposes as the sum of 98 distinct irreducible characters

$$\chi^\lambda = 16\chi + 16\overline{\chi} + 8\sum_{i=1}^{24} \alpha_i + 16\sum_{i=1}^{56} \beta_i + 32\sum_{i=1}^{16} \gamma_i,$$

where $\alpha_i, \beta_i, \gamma_i$ are real-valued with degrees $\alpha_i(1) = 2^{15}$, $\beta_i(1) = 2^{16}$, and $\gamma_i(1) = 2^{17}$.

(4) χ and $\overline{\chi}$ have values of $\pm 256i$, where $i = \sqrt{-1}$, on the pair conjugacy classes of $\mathbf{G}$ which contain elements not conjugate to their inverses. On all other conjugacy classes, χ and $\overline{\chi}$ have real values.

Remark. (1) was conjectured by Isaacs and Karagueuzian in [11].

Proof. Write $\mathbf{G} = \mathbf{U}_{13}(\mathbb{F}_2)$. We construct a decomposition sequence $S = \{(\lambda_i, \mathfrak{h}_i, \mathbf{L}_i, \mathbf{R}_i, \mathfrak{i}_i)\}_{i=0}^4 \in \mathscr{D}(\mathbf{G})$ by the following process. First, let $\lambda_0 = 0 \in \mathfrak{u}_{13}(\mathbb{F}_2)$, and let $\lambda_1 \in \mathfrak{h}_1 = \mathfrak{u}_{13}^*(\mathbb{F}_2)$ be the linear functional corresponding to the set partition $1, 5, 7, 9, 13 \mid 2, 6, 8, 12 \mid 3, 10 \mid 4, 11$; in other words, let $\lambda_1 = e_{1,5} + e_{2,6} + e_{3,10} + e_{4,11} + e_{5,7} + e_{6,8} + e_{7,9} + e_{8,12} + e_{9,13}$. In the notation of Proposition 9.1, the set $\mathcal{I}_{\lambda_1}$ is then given by the positions marked $\blacksquare$ in the following matrix:

$$\underbrace{\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & \blacksquare & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & \blacksquare & \blacksquare & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & 0 & \blacksquare & \blacksquare & 0 & 0 & 0 & \blacksquare & 1 & 0 & 0 \\
 & & & & 0 & \blacksquare & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & \blacksquare & 1 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & \blacksquare & 1 & 0 & 0 & 0 & 0 \\
 & & & & & & & 0 & \blacksquare & \blacksquare & \blacksquare & 1 & 0 \\
 & & & & & & & & 0 & \blacksquare & \blacksquare & \blacksquare & 1 \\
 & & & & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 \\
 & & & & & & & & & & & 0 & 0 \\
 & & & & & & & & & & & & 0
\end{pmatrix}}_{\mathcal{I}_{\lambda_1}}.$$

Here the positions with ones give $\mathrm{supp}(\lambda_1)$. Viewing $\mathfrak{h}_2 = \mathbb{F}_2\text{-span}\{e_{ij} \mid (i,j) \in \mathcal{I}_{\lambda_1}\}$, let $\lambda_2 \in \mathfrak{h}_2$ be the linear functional represented by the matrix $\lambda_2 = e_{3,5} + e_{4,6} + e_{8,10} + e_{9,11}$, and let $\lambda_3 = \mathbf{0} \in \mathfrak{h}_3$ and $\lambda_4 = \mathbf{0} \in \mathfrak{h}_4$. These choices of λ_i uniquely determine the vector spaces $\mathfrak{h}_i, \mathfrak{i}_i$ and groups $\mathbf{L}_i, \mathbf{R}_i$, and the following table lists the relevant properties of the resulting decomposition sequence S:

i	$\lvert\mathfrak{h}_i\rvert$	$\lvert\mathbf{L}_i\rvert$	$\lvert\mathbf{LStab}_S(\lambda_i)\rvert$	$\lvert\mathbf{L}_i \circledast \lambda_i\rvert$	$\lvert\mathfrak{i}_i\rvert$	Number of two-sided orbits in $\lvert\mathfrak{h}_i\rvert$
0	2^{78}	2^{78}	1	2^{78}	2^{78}	1
1	2^{78}	2^{78}	2^{51}	2^{27}	2^{15}	27644437
2	2^{15}	2^{66}	2^{59}	2^{7}	2^{5}	58
3	2^{5}	2^{64}	2^{61}	2^{3}	2	1
4	2	2^{62}	2^{61}	2	2	1

Figure 2: Properties of the degenerate decomposition sequence S

The last line of this table shows that S is degenerate since $|\mathfrak{i}_3| = |\mathfrak{i}_4| = 2 > 1$, and so χ_S is a reducible character which we cannot decompose by our generic constructions. In particular we see that $\langle \chi_S, \chi_S \rangle = |\mathfrak{i}_4| = 2$, and so χ_S has exactly two irreducible constituents. To construct these characters, we observe the following. First, by direction computation we find that $\mathbf{L}_4$ is given by the subgroup $\mathbf{H}$ in the theorem statement, and that the stabilizer subgroup $\mathbf{LStab}_S(\lambda_4)$ is the set of elements in $\mathbf{H}$ with $b = 0$ in the notation used above. Since $\mathbf{LStab}_S(\lambda_4)$ is a subgroup of $\mathbf{L}_4$ of index two, the quotient group $\mathbf{QL}_4 = \mathbf{L}_4/\mathbf{LStab}_S(\lambda_4) \cong \mathbb{Z}_2$ has one non-identity element. Let $g \in \mathbf{L}_4$ be a representative of this nontrivial coset, and note that $g^2 \in \mathbf{LStab}_S(\lambda_4)$. For example, we can take g to be the matrix $g = 1 + e_{3,4} + e_{5,6} + e_{6,7}$. There are two elements of the vector space $\mathfrak{h}_4 \cong \mathbb{F}_2$; we denote these functionals by $\mathbf{0}$ and γ. Now let $v_0, v_1 \in \mathbb{C}\mathbf{G}$ be the vectors $v_0 = v_S(\mathbf{0}) = v_S(\lambda_4)$ and $v_1 = v_S(\gamma) = v_S(g \circledast \mathbf{0})$. One can check by direct computation that $\alpha_S(g, \mathbf{0}) \neq \alpha_S(g, \gamma)$, but this also follows abstractly from fact that if $\alpha_S(g, \mathbf{0}) = \alpha_S(g, \gamma)$ then

$$\mathbb{C}\text{-span}\left\{ \frac{1}{2} \left(\overline{v_0} \pm \overline{v_1} \right) \right\} = \mathbb{C}\text{-span}\left\{ \frac{1}{|\mathfrak{i}_4|} \sum_{X \in \mathfrak{i}_4} \theta(\Gamma(X)) \overline{v_S(\lambda_4 + X)} \right\}$$

would be an $\mathbf{L}_4$-invariant subspace of $W_S = \mathbb{C}\mathbf{L}_4 v_S(\lambda_4)$ for either of the two nontrivial $\mathbb{F}_2$-linear functionals $\Gamma : \mathfrak{i}_4 \to \mathbb{F}_2$. Since V_S is induced from W_S, it follows that we could decompose V_S by extending the sequence S, which contradicts Corollary 8.1.

Given this fact, the vector spaces

$$U_1 = \mathbb{C}\text{-span}\{v_0 + iv_1\} \qquad \text{and} \qquad U_2 = \mathbb{C}\text{-span}\{v_0 - iv_1\}$$

must be $\mathbf{L}_4$-submodules of W_S. In particular, we have

$$g(v_0 \pm iv_1) = \alpha_S(g, \mathbf{0})v_1 \pm i\alpha_S(g, \gamma)v_0 = \alpha_S(g, \mathbf{0})(v_1 \mp iv_0) = \mp i\alpha_S(g, \mathbf{0})(v_0 \pm iv_1).$$

A symmetric argument shows U_1 and U_2 are also right $\mathbf{R}_4$-invariant. Since clearly $W_S = U_1 \oplus U_2$, it follows that V_S decomposes as a direct sum of induced modules $V_S = \mathrm{Ind}_{\mathbf{L}_4}^{\mathbf{G}}(U_1) \oplus \mathrm{Ind}_{\mathbf{L}_4}^{\mathbf{G}}(U_2)$.

Let $\tau, \overline{\tau}$ be the characters of U_1, U_2. These characters are linear and given explicitly by $\tau(x) = \pm i\alpha_S(x, \lambda_4)$ for $x \notin \mathbf{LStab}_S(\lambda_4)$ and $\tau(x) = \alpha_S(x, \lambda_4)$ for $x \in \mathbf{LStab}_S(\lambda_4)$. Now let $\chi = \tau^{\mathbf{G}}$. To show that χ takes non-real values, we evaluate $\tau^{\mathbf{G}}$ on the following element of $\mathbf{L}_4 = \mathbf{H} \subseteq \mathbf{G}$:

$$g = \begin{pmatrix}
1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\
 & & & & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & 1 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 1 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 \\
 & & & & & & & 1 & \mathbf{1} & 0 & 0 & 0 & 0 \\
 & & & & & & & & 1 & 0 & 0 & 0 & \mathbf{1} \\
 & & & & & & & & & 1 & \mathbf{1} & 0 & 0 \\
 & & & & & & & & & & 1 & \mathbf{1} & 0 \\
 & & & & & & & & & & & 1 & \mathbf{1} \\
 & & & & & & & & & & & & 1
\end{pmatrix}.$$

The nonzero positions of g above the diagonal are $(1,2)$, $(2,3)$, $(2,5)$, $(2,6)$, $(3,4)$, $(4,8)$, $(4,9)$, $(4,11)$, $(5,6)$, $(6,7)$, $(7,8)$, $(7,10)$, $(8,9)$, $(9,13)$, $(10,11)$, $(11,12)$, $(12,13)$. One can check using a computer algebra system that the linear equation $xg = g^{-1}x$ has no solution $x \in \mathbf{G}$, and so g is not conjugate to its inverse. More directly, one can construct g by conjugating the element given in [10]. Of course we could just use the latter element; we exhibit g to show that $\mathbf{H}$ contains an element which is not conjugate to its inverse in $\mathbf{G}$.

We evaluate $\chi(g)$ by the formula

$$\chi(g) = \sum_{x \in \hat{\mathbf{G}}} \dot{\tau}(xgx^{-1}), \qquad \dot{\tau}(h) = \begin{cases} \tau(h), & h \in \mathbf{H} \\ 0, & h \notin \mathbf{H} \end{cases}$$

where $\hat{\mathbf{G}}$ denotes a set of representatives of the cosets $\mathbf{G}/\mathbf{H}$. We can take $\hat{\mathbf{G}}$ to be the set

$$\hat{\mathbf{G}} = \left\{ \begin{pmatrix}
1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & 1 & \bullet & \bullet & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & 1 & \bullet & 0 & 0 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & 1 & 0 & 0 & 0 & 0 & \bullet & 0 & \bullet & \bullet & \bullet \\
 & & & & 1 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & 1 & 0 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & 1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & 1 & \bullet & 0 & 0 & \bullet & \bullet \\
 & & & & & & & & 1 & 0 & 0 & 0 & \bullet \\
 & & & & & & & & & 1 & \bullet & \bullet & \bullet \\
 & & & & & & & & & & 1 & \bullet & \bullet \\
 & & & & & & & & & & & 1 & \bullet \\
 & & & & & & & & & & & & 1
\end{pmatrix} \;\middle|\; \bullet \in \mathbb{F}_2 \right\}.$$

Using a computer, one can check directly $gh^{-1} \notin \mathbf{H}$ for $g, h \in \hat{\mathbf{G}}$ with $g \neq h$. Since $\hat{\mathbf{G}}$ has only 2^{16} elements, computing the value of the induced character is quite manageable. The algorithm we implemented yields a value of $\chi(g) = \pm 256\sqrt{-1}$, where the sign depends on an arbitrary assignment of τ to U_1 or U_2. Thus $\chi, \overline{\chi}$ are the complex irreducible characters of $\mathbf{U}_{13}(\mathbb{F}_2)$; they are induced from linear characters of $\mathbf{H} = \mathbf{L}_4$; and their degrees are $\frac{|\mathbf{G}|}{|\mathbf{H}|} = 2^{16}$. By the results in [11], this proves that all real-valued characters of $\mathbf{U}_{13}(\mathbb{F}_2)$ are given by real representations. Finally, (3) in the theorem follows from a separate computation using the code written to calculate the table in Figure 1. □

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