1 Last time: singular value decompositions

The *singular values* of a matrix A are the (nonnegative) square roots of the eigenvalues of the symmetric matrix A^TA. All eigenvalues of A^TA are nonnegative real numbers, so their square roots are defined.

Proposition. If A is a matrix then $\operatorname{rank} A$ is the number of nonzero singular values of A.

Recall that an *orthogonal matrix* is a square matrix U with orthonormal columns.

An orthogonal matrix U is invertible and has $U^{-1} = U^T$.

A *singular value of decomposition* for an $m\times n$ matrix A is an expression $A = U\Sigma V^T$ where U is an $m\times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is the unique $m \times n$ matrix whose entries in positions $(1, 1), (2, 2), \dots, (r, r)$ are the decreasing list of nonzero singular values of A, and which has 0 in all other positions.

Theorem. Every matrix has a singular value decomposition.

2 Final review

The final exam will be cumulative so you should review the topics we covered before the midterm — see Lecture 13. However, all of the topics we've covered since the midterm build on this foundation, so the earlier material shouldn't be too far in the background in any case.

What follows is a sketch of the key concepts from each major topic group in the course. This outline is meant as a baseline for what you should know going into the final examination. It's possible that other things we've seen in class (but which are not mentioned below) might appear on the test. But what's mentioned below is the stuff you should review first.

Pre-midterm topics:

1. **Linear systems and row reduction**.

   Know how to represent a linear system via its coefficient matrix and augmented matrix.

   Know how to reduce a matrix via row operations to its reduced echelon form, in order to determine the pivot columns, free variables, to find and count the solutions of the corresponding linear system, etc.

2. **Vectors and matrices**.

   Know definitions of vectors in $\mathbb{R}^n$ and $m \times n$ matrices. Although sometimes we talk about vectors with complex entries, all matrices are assumed to be arrays of real numbers.

   Know how to draw vectors $v \in \mathbb{R}^2$ as arrows in the plane.

   Know how to add vectors $u + v$, matrices $A + B$, and to rescale vectors and matrices by scalars. Know how to evaluate the matrix-vector product Av when this is defined.

   Important definitions: linear combination, span, linear independence.

   Know definition of zero vector and identity matrix $I = I_n$. Understand that $IA = AI = A$ for any matrix A (of the appropriate size).

   Know definition of the matrix transpose A^T, and that $(AB)^T = B^TA^T$.

   Review different classes of matrices: diagonal, symmetric, orthogonal, upper- and lower-triangular, permutation matrix.

3. **Linear transformations**.

   Know definitions of the domain, codomain, image, and range of a function $f : X \to Y$.

   Know definitions of onto, one-to-one, and invertible functions.

   Understand linearity: a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if $T(u+v) = T(u)+T(v)$ and $T(cv) = cT(v)$ for $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. This implies that $T(0) = 0$.

   Equivalently, a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if there exists an $m \times n$ matrix A such that $T(v) = Av$ for $v \in \mathbb{R}^n$. The matrix A is called the *standard matrix* of T.

   If $e_i \in \mathbb{R}^n$ is the vector with 1 in row i and 0 in all other rows, then the standard matrix of a linear function T is the matrix $A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$.

   Understand when the composition $T \circ U$ of two linear functions T and U is defined. Remember that the standard matrix of $T \circ U$ is the product AB where A is the standard matrix of T and B is the standard matrix of U.

   When $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, remember that T is onto iff the columns of A span $\mathbb{R}^m$, and that T is one-to-one iff the columns of A are linearly independent.

4. **Inverses**.

   Know definition of the inverse of an invertible function. Remember that a linear function $T : \mathbb{R}^n \to \mathbb{R}^m$ is invertible only if $n = m$, but in this case has an inverse which is also linear.

   Know definition of the inverse A^{-1} of a matrix A. Only square matrices are invertible. If A^{-1} and B^{-1} and AB are all defined, then $(AB)^{-1} = B^{-1}A^{-1}$.

   An $n \times n$ matrix A is invertible if and only if one/all of the following equivalent conditions holds:

   (a) The columns of A span $\mathbb{R}^n$.

   (b) The columns of A are linearly independent.

   (c) $\texttt{RREF}(A) = I$.

   (d) $\det A \neq 0$.

   (e) A has no zero singular values.

   Know how to compute A^{-1} by row reducing $\begin{bmatrix} A & I \end{bmatrix}$.

   Remember the formula $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ which is valid when $ad - bc \neq 0$.

5. **Subspaces**.

   Important definitions: subspace, basis, dimension, column space, null space, rank.

   An $n \times n$ matrix A is invertible iff $\operatorname{Col} A = \mathbb{R}^n$ and $\operatorname{Nul} A = \{0\}$. Either condition implies the other.

   The pivot columns of A are a basis for $\operatorname{Col} A$.

   Know how to find a basis for $\operatorname{Nul} A$ by row reduction.

   Rank theorem: if A has n columns then $n = \operatorname{rank} A + \dim \operatorname{Nul} A$ where $\operatorname{rank} A = \dim \operatorname{Col} A$.

   Dimension theorem: if H is a subspace of dimension p, then any p linearly independent vectors in H are a basis for H, and any set of p vectors spanning H is a basis for H.

6. **Determinants**.

   Know various definitions of the determinant of a square matrix A:

   (a) $\det A = \sum_{X \in S_n} (-1)^{\operatorname{inv}(X)} \Pi(X, A)$.

(b) $\det A = A_{11} \det A^{(1,1)} - A_{12} \det A^{(1,2)} + A_{13} \det A^{(1,3)} - \dots$

(c) det is the unique function with $\det I = 1$ and two other properties. (What are they?)

Know how to compute $\det A$ by row reducing, while keeping track of some auxiliary data.

Remember formulas for $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Remember that the determinant of a triangular matrix is the product of its diagonal entries.

Product formula: $\det(AB) = (\det A)(\det B)$ so $\det(A^{-1}) = \frac{1}{\det A}$.

Understand the following statements: Adding one column to another does not change the determinant. Switching two columns multiplies the determinant by -1.

Understand how $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the area of a certain parallelogram.

Post-midterm topics:

7. **Vector spaces**.

   Know definition and examples of vector spaces in general, besides the motivating case of $\mathbb{R}^n$.

   Understand how notations of subspace, linear transformation, linear combination, span, linear independence, basis, and dimension extend to abstract vector spaces.

8. **Eigenvalues and eigenvectors**.

   Be familiar with definition: a vector $v \in \mathbb{R}^n$ is an eigenvector of a matrix A with eigenvalue $\lambda \in \mathbb{R}$ if $Av = \lambda v$ and $v \neq 0$.

   Know how to make sense of this generalisation: a vector $v \in \mathbb{C}^n$ is a (complex) eigenvector of a matrix A with (complex) eigenvalue λ if $Av = \lambda v$ and $v \neq 0$.

   The eigenvalues of a matrix A are the roots of the characteristics polynomial $\det(A - xI)$. These may be complex numbers, and they may be repeated. The sum of these roots (with multiplicity) is the trace $\mathrm{tr}A$. Their product is $\det A$.

   The eigenvalues of a triangular matrix are its diagonal entries.

   Know how to find the eigenvalues of A by factoring $\det(A - xI)$. Given an eigenvalue λ, know how to find a corresponding eigenvector. This can be done by computing a basis for $\mathrm{Nul}(A - \lambda I)$. When this subspace is nonzero, it is called the λ-eigenspace of A.

   Know what the relationship is between the eigenvectors and eigenvalues of A, A^T, and A^{-1}.

9. **Diagonalisation**.

   Square matrices A and B are similar if $A = PBP^{-1}$ for some invertible matrix P. Similar matrices have the same size and the same eigenvalues.

   A matrix A is diagonalisable if it is similar to a diagonal matrix. This can happen only if A is square. An $n \times n$ matrix A is diagonalisable if and only if it has n linearly independent eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$. In this case, it holds that $A = PDP^{-1}$ where $P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$ and D is the diagonal matrix whose entry in position (i, i) is the eignenvalue of v_i.

   A useful shortcut: an $n \times n$ matrix is diagonalisable whenever it has n distinct eigenvalues. Eigenvectors with distinct eigenvalues (of a fixed matrix) are always linearly independent.

   A matrix with repeated eigenvalues may still be diagonalisable. Know how to check if a matrix A is diagonalisable in general: compute the distinct eigenvalues, then compute the dimensions of the corresponding eigenspaces. The matrix is diagonalisable if and only if these dimensions sum to n where A is $n \times n$.

Given a matrix A which is diagonalisable, know how to find a decomposition $A = PDP^{-1}$. Know how to use this to give an exact formula for A^k for any integer k.

10. **Inner products and diagonalisation**.

    Know definitions and properties of the inner product $u \bullet v$, the length $\|v\| = \sqrt{v \bullet v}$, unit vectors, orthogonal vectors, orthonormal vectors.

    Know definition of the orthogonal complement V^T of a subspace $V \subset \mathbb{R}^n$. We have $\dim V + \dim V^\perp = n$ and $V \cap V^\perp = \{0\}$. If A is a matrix then $(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$.

    Orthogonal projections: if $y \in \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ is a subspace, then there is a unique vector $\mathrm{proj}_W(y) \in W$ such that $y - \mathrm{proj}_W(y) \in W^\perp$. This is the orthogonal projection of y onto W.

    Know how to compute $\mathrm{proj}_W(y)$ given an orthogonal basis $v_1, v_1, \ldots, v_p$ for W. Know that the orthogonal projection is the vector $v \in \mathbb{R}^n$ which minimises the length $\|y - v\|$.

    Review the definition of the Gram-Schmidt process. Know how to use it to convert a basis for a subspace to an orthogonal basis, and to compute a QR factorisation of a matrix with linearly independent columns.

11. **Least-squares problems**.

    Know definition of a linear-squares solution to a linear system $Ax = b$: a vector $h \in \mathbb{R}^n$ such that $\|Ah - b\| \leq \|Ax - b\|$ for all $x \in \mathbb{R}^n$.

    Understand why a linear system always has at least one least-squares solution.

    The least-squares solutions to $Ax = b$ are the exact solutions to $A^T Ax = A^T b$. Practice solving some least-squares problems.

    Know when a linear system $Ax = b$ has a unique least-squares solution: when A has linearly independent columns, or equivalently when $A^T A$ is invertible.

    Know how to compute lines of best fit.

    Know how to approximate a function given data using least-squares.

12. **Symmetric matrices**.

    A matrix A is symmetric if $A^T = A$.

    Symmetric matrices have all real eigenvalues and are orthogonally diagonalisable.

    We can write $A = UDU^T = UDU^{-1}$ where D is a diagonal matrix and U is an orthogonal matrix (which means $U^T = U^{-1}$) if and only if A is symmetric.

    If A is any matrix then $A^T A$ is symmetric with nonnnegative eigenvalues.

    The singular values of A are the square roots of the eigenvalues of $A^T A$.

    Know definition and how to construct a singular value decomposition for a matrix A.