Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, February 13**.

Throughout, let n be a positive integer and let $[n] = \{1, 2, \dots, n\}$.

1. Let $\tau \in S_{2n}$ be the permutation mapping $i \mapsto 2n + 1 - i$. Define

$$B_n = \{w \in S_{2n} : w\tau = \tau w\}.$$

   Prove that B_n is a subgroup of S_{2n} and find a formula for $|B_n|$.

2. For $w \in B_n \subset S_{2n}$, define $\mathrm{Inv}_B(w)$ as the set of integer pairs $(i, j) \in [2n] \times [2n]$ such that either $i < j$ and $w(i) > w(j)$, or $i = j \leq n < w(i)$. Define $t_1, t_2, \dots, t_n \in B_n$ as the elements

$$t_i = s_i s_{2n-i} \quad \text{for } i \in [n-1] \qquad \text{and} \qquad t_n = s_n$$

   where $s_i = (i, i+1) \in S_{2n}$ for $i \in [2n-1]$.

   (a) Prove that $\mathrm{Inv}_B(w) = \varnothing$ for $w \in B_n$ if and only if $w = 1$.

   (b) Prove that if $i \in [n]$ and $w \in B_n$ then $|\mathrm{Inv}_B(wt_i)| - |\mathrm{Inv}_B(w)| \in \{-2, 2\}$.

3. Define $\mathrm{Des}_B(w)$ for $w \in B_n$ as the set of $i \in [n]$ such that $|\mathrm{Inv}_B(wt_i)| < |\mathrm{Inv}_B(w)|$.

   (a) Prove that $\mathrm{Des}_B(w) = \varnothing$ if and only if $w = 1$.

   (b) Prove that $\mathrm{Des}_B(w) = [n]$ if and only if $w = \tau$.

4. Define $\ell_B(w) = \frac{1}{2}|\mathrm{Inv}_B(w)|$ for $w \in B_n$. Combine the previous two exercises to prove that

$$\{\ell_B(w) : w \in B_n\} = \{0, 1, 2, \dots, n^2\} \qquad \text{and} \qquad B_n = \langle t_1, t_2, \dots, t_n \rangle.$$

5. You have n objects numbered from 1 to n. Assume object i has weight w_i and that $w_1 < w_2 < \cdots < w_n$. You have a scale which can compare two objects and tell you which one is heavier. This scale is a little unreliable, however: each time it is used, there is a small, fixed probability $0 < p < 1$ that the scale will tell you the wrong answer about the relative weights of two objects.

   Suppose you use the scale to compare all $\binom{n}{2}$ pairs of objects. From these independent measurements, construct X as the set of pairs (i, j) with $1 \leq i < j \leq n$ for which the scale reported (correctly) that object i is lighter than object j. Show that for any permutation $w \in S_n$, the conditional probability that $X = \mathrm{Inv}(w)$ given that X is the inversion set of some permutation is

$$\frac{\theta^{\ell(w)}}{(1+\theta)(1+\theta+\theta^2)(1+\theta+\theta^2+\theta^3)\cdots(1+\theta+\theta^2+\cdots+\theta^{n-1})}$$

   where $\theta = p^{-1} - 1$.