Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, February 20**.

1. Show that if Φ is a root system with rank 2 (i.e., every simple system in Φ has size 2) then the associated reflection group $W = \langle s_\alpha : \alpha \in \Phi \rangle$ is isomorphic to a dihedral group.

2. Let $V = \mathbb{R}^n$ with the standard bilinear form, and write $e_1, e_2, \dots, e_n$ for the standard orthonormal basis. Define $<$ as the total order on V with $\sum_{i=1}^n a_i e_i < \sum_{i=1}^n b_i e_i$ if for some index j it holds that $a_j < b_j$ and $a_i = b_i$ for $1 \leq i < j$.

   (a) Let $\Phi = \{\alpha \in \mathbb{Z}^n : (\alpha, \alpha) = 2\}$. Show that Φ is a root system. What is $|\Phi|$?

   (b) Identify the sets of positive and simple roots $\Delta \subset \Pi \subset \Phi$ with respect to the given total order.

   (c) Let $W = \langle s_\alpha : \alpha \in \Phi \rangle$. Prove that when $n \geq 4$ there is an exact sequence

   $$1 \to W \to B_n \to S_2 \to 1.$$

   (d) What is W when $n \in \{1, 2, 3\}$?

3. Let $\Phi \subset V$ be a root system with positive system $\Pi \subset \Phi$ and simple system $\Delta \subset \Pi$. Assume Δ is a basis for V and define ht as the linear map $V \to \mathbb{R}$ with $\mathrm{ht}(\alpha) = 1$ for all $\alpha \in \Delta$. Prove that if $\beta \in \Pi \setminus \Delta$, then $\mathrm{ht}(\beta) > 1$.

4. Let Φ be a root system and suppose $\Delta \subset \Phi$ is a simple system. Let $\Delta' \subsetneq \Delta$ be a proper subset and define $G = \langle s_\alpha : \alpha \in \Delta \rangle$ and $H = \langle s_\alpha : \alpha \in \Delta' \rangle$. Prove that $|H| < |G|$.

5. Show that there are two root systems Φ and Φ' of different ranks whose associated reflection groups $W = \langle s_\alpha : \alpha \in \Phi \rangle$ and $W' = \langle s_\alpha : \alpha \in \Phi' \rangle$ are isomorphic groups.