Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, February 27**.

First, let V be a finite-dimensional real vector space with a symmetric positive definite bilinear form $(\cdot,\cdot)$; let $\Phi \subset V$ be a root system with positive system Π and simple system $\Delta \subset \Pi$; and write $W = \langle s_\alpha : \alpha \in \Phi\rangle$ for the associated reflection group.

1. Define $D = \{v \in V : (v, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}$.
   (a) Show that if $\lambda \in V$ then there exists $w \in W$ such that $\mu = w\lambda \in D$.
   (b) Suppose $\lambda, \mu \in D$ and $w \in W$ are such that $w\lambda = \mu$. Show that $\lambda = \mu$ and that w is a product of reflections fixing λ.
   (c) Show that if $\lambda \in V$ then the subgroup $H = \{w \in W : w\lambda = \lambda\}$ is generated by the reflections it contains.

Now let (W, S) be a Coxeter system and write $m(s,t)$ for the order of st for $s, t \in S$. Recall that this means that W has the presentation $W = \langle s \in S : (st)^{m(s,t)}\rangle$. Assume S is a finite set.

2. Let $\mathrm{sgn} : W \to \{-1, 1\}$ be the unique homomorphism with $\mathrm{sgn}(s) = -1$ for all $s \in S$, and let $W^+ = \ker \mathrm{sgn}$. Suppose the distinct elements of S are $s_1, s_2, \ldots, s_n$. Prove that W^+ is generated by $s_i s_n$ for $1 \leq i \leq n-1$.
3. Prove that if $|S| = n$ and $m(s,t)$ is even for all $s \neq t$ in S then $|W| \geq 2^n$.
4. Prove that every element of W has a unique reduced expression if and only if $m(s,t) = \infty$ for every $s \neq t$ in S.
5. Prove that $s, t \in S$ are conjugate in W if and only if there are elements $s = s_1, s_2, \ldots, s_k = t$ in S for which each $s_i s_{i+1}$ has finite odd order.