Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, March 6**.

Let (W,S) be a Coxeter system with geometric representation $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$. Let $m(s,t)$ be the order of $st \in W$ for $s,t \in S$. Define $\Phi = \{w\alpha_s : w \in W,\ s \in S\}$ and write Φ^+ and Φ^- for the subsets of vectors in Φ which are nonnegative and nonpositive linear combinations of α_s for $s \in S$.

1. Given a reduced expression $w = s_1 \cdots s_r$ ($s_i \in S$), set $\alpha_i = \alpha_{s_i}$ and $\beta_i = s_r s_{r-1} \cdots s_{i+1} \alpha_i$, interpreting β_r as α_r. Prove that $\beta_1, \dots, \beta_r$ are the distinct elements of the set $\{\alpha \in \Phi^+ : w\alpha \in \Phi^-\}$.
2. If W is infinite, prove that its length function takes arbitrarily large values. Prove in this case that $-1 \in \mathrm{GL}(V)$ does not belong to the image of W under its geometric representation. If W is finite prove that there is a unique element in W of maximum length, which maps Φ^+ to Φ^-.
3. Let $s,t \in S$ and $w \in W$. Suppose $\ell(ws) = \ell(wt) = \ell(w) - 1$. Prove that $m(s,t) < \infty$ and $\ell(wv_0) = \ell(w) - m(s,t)$ for $v_0 = stst \cdots = tsts \cdots$ (both expressions with $m(s,t)$ factors).
4. A *braid relation* is a transformation (between two finite products of elements of S) of the form

$$r_1 \cdots r_i \underbrace{stststs\cdots}_{m(s,t)\text{ factors}} r_{i+1} \cdots r_n \ \leftrightarrow\ r_1 \cdots r_i \underbrace{tststst\cdots}_{m(s,t)\text{ factors}} r_{i+1} \cdots r_n$$

   where $r_i, s, t \in S$ and $1 < m(s,t) < \infty$. Note that such a transformation does not change the value of the product of these generators in W, since $(st)^{m(s,t)} = 1$.

   (a) Let $w \in W$. Prove that if we are given any two reduced expressions for w, then one can be transformed to the other by applying a sequence of braid relations.

   (b) Prove that an arbitrary expression $w = s_1 \cdots s_r$ ($s_i \in S$) is not reduced if and only if it is possible to apply a sequence of braid relations to this expression to obtain a new expression $w = t_1 \cdots t_r$ ($t_i \in S$) in which $t_i = t_{i+1}$ for some i.
5. Prove the following assertions directly by exhibiting an element of W of infinite order:

   (a) W is infinite if its Coxeter graph contains a cycle.

   (b) W is infinite if its Coxeter graph is connected and contains two edges whose weights are ≥ 4.

   (c) W is infinite if there exists $s \in S$ such that $\{t \in S : m(s,t) > 2\}$ has size ≥ 4.