Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, March 13**.

Let (W, S) be a Coxeter system with geometric representation $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$. Write $\ell : W \to \mathbb{N}$ for the usual length function and $<$ for the Bruhat order on W.

1. Let $J \subset S$. Prove that W_J is normal in W if and only if each $s \in S - J$ commutes with all $t \in J$.
2. Suppose $w = w^{-1} \in W$ and $s \in S$. Show that if $\ell(ws) < \ell(w)$, then $\ell(sws) = \ell(w) - 2$ or $w\alpha_s = -\alpha_s$.
3. Prove that if W is finite with longest element w_0 then $u < v$ if and only if $w_0 u w_0 < w_0 v w_0$ for $u, v \in W$.
4. Let $I, J \subset S$. Show if $w \in W$ then the double coset $W_I w W_J = \{awb : a \in W_I,\ b \in W_J\}$ contains a unique element of minimal length.
5. An antichain in a partially ordered set is a set of elements, no two of which are comparable. Prove that if S is finite then every antichain in the Bruhat order of W is finite.