Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, March 20**.

Let (W, S) be a Coxeter system with geometric representation $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$. Let V^* be the dual space of $\mathbb{R}$-linear map $V \to \mathbb{R}$. Recall that

$$C = \{f \in V^* : f(\alpha_s) > 0 \text{ for all } s \in S\} \qquad \text{and} \qquad D = \{f \in V^* : f(\alpha_s) \geq 0 \text{ for all } s \in S\}$$

and that the Tits cone of (W, S) is the convex cone $U = \bigcup_{w \in W} w(D)$.

1. Prove that if $U = V^*$ then W is finite.

   Hint: Find $w \in W$ such that $w(D) \cap (-C) \neq \varnothing$. Then deduce that $w^{-1}\alpha_s < 0$ for all $s \in S$.

2. Assume (W, S) is irreducible and write $Z(W) = \{w \in W : wg = gw \text{ for all } g \in W\}$ for the center of W. Prove that if W is finite with longest element w_0, and w_0 acts on V as the scalar -1, then $Z(W) = \{1, w_0\} \cong \mathbb{Z}_2$, and that otherwise $Z(W) = \{1\}$.

3. Let $\rho : G \to \mathrm{GL}(E)$ be an irreducible representation of a group G, with E a finite-dimensional vector space over $\mathbb{C}$. An element $g \in G$ is a *pseudo-reflection* if g has finite order and its 1-eigenspace in E (that is, $\{e \in E : \rho(g)e = e\}$) has codimension 1. Show that if G contains at least one pseudo-reflection then the only endomorphisms of E commuting with $\rho(G)$ are the scalars.

4. Prove that if W is crystallographic relative to its geometric representation, then every parabolic subgroup W_J for $J \subset S$ is also crystallographic relative to its geometric representation. (Recall that W is crystallographic relative to its geometric representation if there exists a lattice in V, i.e., a free $\mathbb{Z}$-module V, stabilized by W.)

5. Suppose $n \geq 2$ integers with positive sum are arranged in a circle. The following game is played. If at least one number is negative, then the player may pick a negative number, add it to its neighbors, and reverse its sign. The game terminates when all the numbers are nonnegative. Prove that the game terminates in the same number of steps and in the same final position no matter how it is played.