

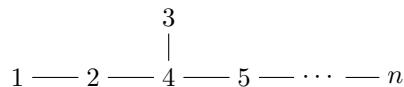
Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Wednesday, March 29**.

Let (W, S) be a Coxeter system with geometric representation $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$. Assume $|S| = n < \infty$. Write $S = \{s_1, s_2, \dots, s_n\}$ and define the $n \times n$ matrix $C = [2(\alpha_{s_i}, \alpha_{s_j})]_{i,j \in [n]}$. In other words, C is twice the usual matrix of the bilinear form (\cdot, \cdot) on V .

1. Find a formula for $\det C$ in the following cases:
 - (a) (W, S) has type A_n for $n \geq 1$.
 - (b) (W, S) has type B_n for $n \geq 2$.
 - (c) (W, S) has type D_n for $n \geq 4$.
 - (d) (W, S) has type E_n for $n \geq 6$, so that the corresponding Coxeter graph is



Use these results to show that (\cdot, \cdot) is positive definite in types A_n, B_n, D_n, E_6, E_7 , and E_8 .

2. Prove that $\det C = 0$ if (W, S) has type $\tilde{A}_n, \tilde{B}_n, \tilde{C}_n$, or \tilde{D}_n .
3. Let A be an $n \times n$ real symmetric matrix. Assume $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$. Assume also that A is *indecomposable*, meaning that we cannot reorder the standard basis of \mathbb{R}^n to write A in block diagonal form; more precisely, assume that we cannot partition $[n]$ into two disjoint sets I, J such that $A_{ij} = 0$ whenever $i \in I$ and $j \in J$. Finally, assume $A_{ij} \leq 0$ for all $i \neq j$ in $[n]$.
 - (a) Let $v \in \mathbb{R}^n$. Show that $v^T A v = 0$ if and only if $Av = 0$.
 - (b) Let $v \in \mathbb{R}^n$ and define $|v|$ is the vector given by replacing all entries of v by their absolute values. Show that if $Av = 0$ then $A|v| = 0$.
 - (c) Use the previous part and the indecomposability of A to deduce that if $0 \neq v \in \mathbb{R}^n$ and $Av = 0$ then $v_i \neq 0$ for all $i \in [n]$. Conclude that the nullspace of A has dimension at most one.
 - (d) Show that if λ is the smallest eigenvalue of A , then it occurs with multiplicity one and has an eigenvector with all positive coordinates. *Hint: consider the matrix $A - \lambda I$.*
4. Let Γ be a connected Coxeter graph of positive type with n vertices labeled $1, 2, \dots, n$. Let A be the matrix of the associated bilinear form on \mathbb{R}^n . Suppose Γ' is a proper subgraph of Γ on the vertices $1, 2, \dots, k$. (Here, subgraph means that we takes a subset of the vertices and any subset of the edges on the those vertices.) Let A' be the matrix of the bilinear form on \mathbb{R}^k associated to Γ' . Use the previous exercise to prove that $v^T A' v > 0$ for all $0 \neq v \in \mathbb{R}^k$.
5. Fix an integer $n \geq 2$. Let W be the set of bijections $w : \mathbb{Z} \rightarrow \mathbb{Z}$ with

$$w(i+n) = w(i) + n \text{ for all } i \in \mathbb{Z} \quad \text{and} \quad w(1) + w(2) + \cdots + w(n) = 1 + 2 + \cdots + n.$$

Let $S = \{s_1, s_2, \dots, s_n\}$ where $s_i \in W$ is the map $\mathbb{Z} \rightarrow \mathbb{Z}$ with

$$s_i(j) = \begin{cases} j+1 & \text{if } i \equiv j \pmod{n} \\ j-1 & \text{if } i+1 \equiv j \pmod{n} \\ j & \text{otherwise.} \end{cases}$$

Prove that (W, S) is a Coxeter system, and identify its Coxeter graph.