Instructions: Complete the following exercises.

Your work on the assigned problems will be graded on clarity of exposition as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due in class on **Monday, April 10**.

1. Fix a positive integer n and let (W,S) be the Coxeter system with $W = S_n$ and $S = \{s_1, s_2, \dots, s_{n-1}\}$ where $s_i = (i, i+1) \in S_n$. Let $\mathbb{Q}(x)$ be the field of rational functions over the rational numbers in an indeterminate x. Let $\mathcal{H} = \mathbb{Q}(x)\text{-span}\{T_w : w \in W\}$ be the generic Hecke algebra of (W,S) over $\mathbb{Q}(x)$ with structure constants $a_s = x-1$ and $b_s = x$ for all $s \in S$, so that

$$T_u T_v = T_{uv} \text{ if } \ell(uv) = \ell(u) + \ell(v) \qquad \text{and} \qquad T_s^2 = (x-1)T_s + xT_1 \text{ for } s \in S.$$

Define $T_i = T_{s_i}$ for $i \in [n-1]$, and let V be the free $\mathbb{Q}(x)$-module with basis $e_1, e_2, \dots, e_n$.

   (a) Show that V has an $\mathcal{H}$-module structure in which

$$T_i e_j = \begin{cases} x e_j & \text{if } j \notin \{i, i+1\} \\ e_{i+1} & \text{if } j = i \\ x e_i + (x-1) e_{i+1} & \text{if } j = i+1 \end{cases} \qquad \text{for } i \in [n-1],\ j \in [n].$$

   (b) Prove that $V = V' \oplus V''$ as an $\mathcal{H}$-module where V' affords a 1-dimensional representation of $\mathcal{H}$ and V'' is the free $\mathbb{Q}(x)$-module spanned by $\{xe_1 - e_2, xe_2 - e_3, \dots, xe_{n-1} - e_n\}$.

   (c) Determine how each T_i acts on V'' and show that V'' is a simple $\mathcal{H}$-module.

2. Let G be a finite group and let $B \subset G$ be a subgroup. Let $e = \frac{1}{|B|}\sum_{b \in B}$ and $\mathcal{H}(G,B) = e\mathbb{C}Ge$. Let M be the right G-module $e\mathbb{C}G$ and define $\mathrm{End}_{\mathbb{C}G}(M)$ as the algebra of linear maps $\lambda : M \to M$ with $\lambda(mg) = \lambda(m)g$ for all $m \in M$ and $g \in G$. Recall that the map $\phi : \mathrm{End}(M) \to \mathcal{H}(G,B)$ given by $\phi(\lambda) = \lambda(e)$ is an isomorphism of $\mathbb{C}$-algebras.

   (a) Define $t_B : \mathbb{C}G \to \mathbb{C}$ as the linear map with $t_B(1) = |B|$ and $t_B(g) = 0$ for $g \in G - \{1\}$. Prove that $t_B(xy) = t_B(yx)$ for all $x, y \in \mathbb{C}G$.

   (b) Show that there is a constant $r \in \mathbb{C}$ such that $\mathrm{tr}(\lambda) = r \cdot t_B(\phi(\lambda))$ for all $\lambda \in \mathrm{End}(M)$, where tr denotes the usual trace of a linear map $M \to M$. What is the value of r?

   (c) Show that if $x \in \mathcal{H}(G,B)$ is nilpotent, then $t_B(x) = 0$.

   (d) Show that if $x \in \mathcal{H}(G,B)$ is nonzero, then there exists $a \in \mathcal{H}(G,B)$ such that $t_B(ax) \neq 0$.

   (e) Show that $\mathcal{H}(G,B)$ has no nilpotent left or right ideals, and is therefore semisimple.

3. Let (W,S) be a Coxeter system, let A be a commutative ring, and let $\mathcal{H}$ be a generic Hecke algebra of (W,S) over A with structure constants $a_s, b_s \in A$ for $s \in S$, so that $T_s^2 = a_s T_s + b_s T_1$.

   (a) Suppose $w = s_1 s_2 \cdots s_n = t_1 t_2 \cdots t_n$ are two reduced expressions for $w \in W$. Show that

$$b_{s_1} b_{s_2} \cdots b_{s_n} = b_{t_1} b_{t_2} \cdots b_{t_n}.$$

   Thus, we can define $b_w = b_{s_1} \cdots b_{s_n} \in A$ when $w = s_1 \cdots s_n \in W$ is any reduced expression.

   (b) Define $\tau : \mathcal{H} \to A$ as the linear map with $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. Prove that

$$\tau(T_u T_v) = \begin{cases} b_u & \text{if } u^{-1} = v \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } u, v \in W.$$

4. Let (W,S) be an arbitrary Coxeter system, let $\mathcal{H}$ be the generic Hecke algebra of (W,S) over $\mathbb{Z}[x]$ with structure constants $a_s = x-1$ and $b_s = x$ for all $s \in S$. Determine all 1-dimensional representations of $\mathcal{H}$ and, when W is finite, find left submodules of $\mathcal{H}$ which afford these representations.