1 Course details.

Coxeter systems and their associated Iwahori-Hecke algebras are central to a lot active research at the intersection of combinatorics, geometry and representation theory. The goal of this course is to introduce the basic theory of these objects, highlight some interesting applications, and discuss current open problems.

- Books:

  (a) *Reflection groups and Coxeter groups* by Humphreys

  (b) *Combinatorics of Coxeter groups* by Bjorner and Brenti

  (c) *Characters of finite Coxeter groups and Iwahori-Hecke algebras* by Geck and Pfeiffer

  The lectures in the first part of the course will mostly follow (a), but (b) and (c) are also excellent references, with slightly different focuses.

- Prerequisites:

  Abstract algebra, from a course like MATH 5111.

  Familiarity with representation theory and combinatorics helpful but not necessary.

- Outline:

  Cover most of Chapters 1, 5, and 7 of Humphreys's book

  Complements in Chapters 2, 3, 4, and 6.

  Discuss related topics, open problems, and research directions in second half, as time allows.

- Grades:

  Grades will be based on problem sets which will be assigned every 1-2 weeks.

2 Motivation: Coxeter theory of the symmetric group

Today's lecture highlights some nice properties of the symmetric group, which we will generalize in subsequent lectures to finite reflection groups and, later, to Coxeter groups.

Let n be a positive integer and define $[n] = \{1, 2, 3, \dots, n\}$. Recall that the *symmetric group* is the group S_n of bijections $[n] \to [n]$. Elements of S_n are called *permutations*.

A *simple transposition* is a permutation of the form $(i, i+1)$, i.e., that maps

$$i \mapsto i+1 \qquad \text{and} \qquad i+1 \mapsto i \qquad \text{and} \qquad j \mapsto j \text{ for } j \in [n] - \{i, i+1\}.$$

Write $s_i = (i, i+1)$ for $i \in [n-1]$ to denote the simple transposition interchanging i and $i+1$.

The *one-line representation* of $w \in S_n$ is the word $w_1 w_2 w_3 \cdots w_n$ where $w_i = w(i)$. For example, 31254 is permutation $w \in S_5$ with $w(1) = 3$, $w(2) = 1$ $w(3) = 2$, $w(4) = 5$ and $w(5) = 4$.

Multiplication in S_n is composition of functions: if $v, w \in S_n$, then $vw : i \mapsto v \circ w(i) = v(w(i))$.

Fact. If $w = w_1 w_2 w_3 \cdots w_n \in S_n$ and $i \in [n-1]$ then $ws_i = w_1 \cdots w_{i-1} w_{i+1} w_i w_{i+2} \cdots w_n$.

A pair of integers (i, j) with $1 \leq i < j \leq n$ is an *inversion* of $w \in S_n$ if $w(i) > w(j)$. Write $\mathrm{Inv}(w)$ for the set of inversions of w. Example: $\mathrm{Inv}(31254) = \{(1, 2), (1, 3), (4, 5)\}$.

Define $\mathrm{Des}_R(w) = \{i : (i, i+1) \in \mathrm{Inv}(w)\}$. Elements of this set are *(right) descents* of w.

Fact. If $i \in [n-1]$ and $w \in S_n$ and $i \notin \mathrm{Des}_R(w)$, then $i \in \mathrm{Des}_R(ws_i)$.

Lemma. Let $w \in S_n$. Then $\mathrm{Des}_R(w) = \varnothing$ if and only if $w = 1$.

Proof. If $w = 1$ then clearly $\mathrm{Inv}(w) = \mathrm{Des}_R(w) = \varnothing$. If $\mathrm{Des}_R(w) = \varnothing$, conversely, then we must have $w_1 < w_2 < \cdots < w_n$. The only way this is possible is if $w = 123 \cdots n = 1$. □

Lemma. If $i \in \mathrm{Des}_R(w)$ then $(j, k) \mapsto (s_i(j), s_i(k))$ is a bijection $\mathrm{Inv}(v) - \{(i, i + 1)\} \to \mathrm{Inv}(ws_i)$.

Proof. This follows since $ws_i = w_1 \cdots w_{i-1}w_{i+1}w_iw_{i+2} \cdots w_n$. (Try working through the details!) □

Corollary. If $i \in \mathrm{Des}_R(w)$ then $|\mathrm{Inv}(ws_i)| = |\mathrm{Inv}(w)| - 1$.

If X is a set contained in a group G, then we write $\langle X\rangle$ for the intersection of all subgroups $H \subset G$ with $X \subset H$. One refers to $\langle X\rangle$ as the group *generated* by X.

If $G = \langle X\rangle$ then G is *generated* by X. This occurs if and only if each $g \in G$ is a product of a finite number of elements, each of which either belongs to X or has its inverse in X.

Proposition. $S_n = \langle s_1, s_2, \ldots, s_{n-1}\rangle$.

Proof. Note that $s_i^2 = 1$. Let $w \in S_n - \{1\}$. It suffices to check that w can be written as a product of simple transpositions. We argue by induction on $|\mathrm{Inv}(w)|$. Since $w \neq 1$, the set $\mathrm{Des}_R(w)$ is nonempty. Let $i \in \mathrm{Des}_R(w)$. Then $|\mathrm{Inv}(ws_i)| < |\mathrm{Inv}(w)|$, so by induction we assume that $ws_i \in \langle s_1, s_2, \ldots, s_{n-1}\rangle$, so clearly $w = (ws_i)s_i \in \langle s_1, s_2, \ldots, s_{n-1}\rangle$. □

Thus S_n is generated by simple transpositions, which suggests a natural question: how many simple transpositions must we multiply together to produce a given permutation $w \in S_n$? The following corollary shows that this must be at least the number of inversions of w.

Define the *length* of $w \in S_n$ to be $\ell(w) = |\mathrm{Inv}(w)|$.

Corollary. If $w \in S_n$ and $i \in [n - 1]$ then $\ell(ws_i) = \begin{cases} \ell(w) - 1 & \text{if } i \in \mathrm{Des}_R(w) \\ \ell(w) + 1 & \text{otherwise.} \end{cases}$.

Proof. The second case follows from the first since if $i \notin \mathrm{Des}_R(w)$ then $i \in \mathrm{Des}_R(ws_i)$. □

Example. Let $w_0 = n \cdots 321 \in S_n$ so that $w_0(i) = n + 1 - i$ for $i \in [n]$. Then

$$\mathrm{Inv}(w_0) = \{(i, j) : 1 \leq i < j \leq n\}$$

so $\ell(w_0) = \binom{n}{2}$. Therefore w_0 is the "longest" permutation in S_n, as $\mathrm{Inv}(v) \subset \mathrm{Inv}(w_0)$ for all $v \in S_n$.

Note, the inversion set of a permutation uniquely determines it.

Proposition. If $v, w \in S_n$ and $\mathrm{Inv}(v) = \mathrm{Inv}(w)$ then $v = w$.

Proof. If $\mathrm{Inv}(v) = \mathrm{Inv}(w) = \varnothing$ then $v = w = 1$. Otherwise $\mathrm{Inv}(w) = \mathrm{Inv}(w) \neq \varnothing$, so $\mathrm{Des}_R(v) = \mathrm{Des}_R(w) \neq \varnothing$. If $i \in \mathrm{Des}_R(v) = \mathrm{Des}_R(w)$ then $\mathrm{Inv}(vs_i) = \mathrm{Inv}(ws_i)$ by earlier lemma, and $\ell(vs_i) = \ell(ws_i) < \ell(v) = \ell(w)$, so by induction $vs_i = ws_i$ and therefore $v = (vs_i)s_i = (ws_i)s_i = w$. □

Corollary. S_n contains a unique longest element w_0 of length $\binom{n}{2}$.

A *reduced word* for $w \in S_n$ is a sequence of integers $(i_1, i_2, \ldots, i_k)$ of minimal possible length k such that $w = s_{i_1}s_{i_2} \cdots s_{i_k}$. Let $\mathcal{R}(w)$ be the set of reduced words for $w \in S_n$. For example, $\mathcal{R}(321) = \{(1, 2, 1), (2, 1, 2)\}$. To understand this set, we appeal to the following crucial theorem.

Theorem. Suppose $i_1, i_2, \dots, i_m \in [n-1]$ are such that $w = s_{i_1} s_{i_2} \cdots s_{i_m} \in S_n$. If $\ell(w) < m$, then there are two indices $1 \leq j < k \leq m$ such that $w = s_{i_1} \cdots \widehat{s_{i_j}} \cdots \widehat{s_{i_k}} \cdots s_{i_m}$ where $\widehat{\ }$ means we omit that factor.

We give a slightly tricky direct proof. Later we will see a more general argument.

Proof. First let $k \in [m]$ be minimal such that $\ell(s_{i_1} s_{i_2} \cdots s_{i_k}) \neq k$. Then $\ell(s_{i_1} s_{i_2} \cdots s_{i_{k-1}}) = k-1$, so $\ell(s_{i_1} s_{i_2} \cdots s_{i_k}) = k-2$. Note that $k > 1$, let $v = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}$, and define $b = v(i_k)$ and $a = v(i_k+1)$. Observe that $a < b$. Now let $j \in [m]$ be maximal such that a appears to the left of b in the one-line representation of $s_{i_1} s_{i_2} \cdots s_{i_{j-1}}$ (where if $j = 1$ then this product is interpreted as the identity permutation). Since a is left of b at the outset, such an index j exists and $j < k$. The clever part of the argument has two steps: (1) By the maximality of j it must hold that a and b are adjacent in positions i_j and i_j+1 of the line-line representation of $s_{i_1} s_{i_2} \cdots s_{i_{j-1}}$. (2) It follows that $s_{i_1} \cdots s_{i_{k-1}} = s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_k}$. To deduce these claims, try to visualize the relative positions of a and b in the one-representations of the successive permutations 1, s_{i_1}, $s_{i_1} s_{i_2}$, $s_{i_1} s_{i_2} s_{i_3}$, as so on. Multiplying (2) by $s_{i_k} s_{i_{k+1}} \cdots s_{i_m}$ gives

$$s_{i_1} s_{i_2} \cdots s_{i_m} = s_{i_1} \cdots \widehat{s_{i_j}} \cdots \widehat{s_{i_k}} \cdots s_{i_m}.$$

□

The theorem has several notable corollaries.

Corollary. Every reduced word for $w \in S_n$ has length $\ell(w)$.

Proof. If $(i_1, i_2, \dots, i_k) \in \mathcal{R}(w)$ then $\ell(w) \leq k$ since every time we multiply on the right by a simple transposition the number of inversions goes up by at most one. But we cannot have $\ell(w) < k$ since then the theorem would imply that we could omit two factors from our word without changing its product, contradicting the definition of "reduced." □

Corollary. If $w \in S_n$ then $\ell(w) = \ell(w^{-1})$.

This could also be shown directly. (What is the relationship between $\mathrm{Inv}(w)$ and $\mathrm{Inv}(w^{-1})$?)

Proof. Reversing words is a bijection $\mathcal{R}(w) \to \mathcal{R}(w)^{-1}$, so all words in either set have length $\ell(w)$. □

Corollary (Exchange principle)**.** If $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ and $j \in \mathrm{Des}_R(w)$ then

$$w s_j = s_{i_1} \cdots \widehat{s_{i_k}} \cdots s_{i_m}$$

for some $k \in [m]$.

Proof. By the theorem, we can omit two factors from $s_{i_1} s_{i_2} \cdots s_{i_m} s_j$ without changing the resulting product. If we don't omit s_j, then repeat the process until we do (then add back in the other factors). □

Corollary. The map $\mathrm{sgn} : S_n \to \{\pm 1\}$ given by $\mathrm{sgn}(w) = (-1)^{\ell(w)}$ is a homomorphism.

Proof. It suffices to check that the parities (i.e., value modulo 2) of $\ell(vw)$ and $\ell(v) + \ell(w)$ are equal for all $v, w \in S_n$. This follows since if we concatenate reduced words for v and w, then the theorem implies that we can obtain a reduced word for vw by omitting an even number of factors. □

Most of the definitions and properties we've seen today have natural generalizations to all finite *reflection groups*, which we will define and begin to classify next time.