1 Course updates

The website is up at `http://www.math.ust.hk/~emarberg/Math6150F` and the first homework assignment have been posted. The first assignment is due in class next Monday, 13 February.

2 Reflection groups

Let V be a vector space over the real numbers $\mathbb{R}$, with a bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ that is both *symmetric* and *positive definite*. Recall that these properties are equivalent to requiring that $(u, v) = (v, u)$ for all $u, v \in V$ and $(v, v) > 0$ if $v \in V$ is nonzero.

Two vectors $u, v \in V$ are *orthogonal* if $(u, v) = 0$.

The important thing about this setup is the following basic fact from linear algebra: if $U \subset V$ is any subspace and $U^\perp = \{v \in V : (u, v) = 0 \text{ for all } u \in U\}$ then $U^\perp$ is also a subspace and $V = U \oplus V^\perp$. If U is a one-dimensional space (i.e., a line), then $U^\perp$ is the *hyperplane* orthogonal to U.

Example. If $V = \mathbb{R}^n$ then the standard choice for $(\cdot,\cdot)$ is given by setting $(\sum_i a_i e_i, \sum_i b_i e_i) = \sum_i a_i b_i$.

Definition. The *reflection* through a nonzero vector $\alpha \in V$ is the linear map

$$s_\alpha : v \mapsto v - 2\tfrac{(v,\alpha)}{(\alpha,\alpha)}\alpha \qquad \text{for } v \in V.$$

Note that we don't have to worry about dividing by (α, α) since the form is positive definite.

Fix a nonzero vector $\alpha \in V$. To get some geometric intuition for what s_α does, consider the following simple facts. Each of these statements follows immediately from the definition of s_α.

Lemma. $s_\alpha = s_{c\alpha}$ for any nonzero scalar $c \in \mathbb{R}$.

Lemma. $s_\alpha(\alpha) = -\alpha$.

Lemma. If $v \in V$ and $(v, \alpha) = 0$ then $s_\alpha(v) = v$.

Thus s_α negates α and fixes every vector orthogonal to α. In other words, s_α acts on V by reflecting vectors across the hyperplane orthogonal to α.

Lemma. $s_\alpha^2 = 1$.

Proof. This is clear from the geometric description of s_α just given. Algebraically, we have $s_\alpha^2(v) = s_\alpha(s_\alpha v) = s_\alpha\left(v - 2\tfrac{(v,\alpha)}{(\alpha,\alpha)}\alpha\right) = v - \tfrac{2(v,\alpha)}{(\alpha,\alpha)}\alpha + \tfrac{2(v,\alpha)}{(\alpha,\alpha)}\alpha = v$ for all $v \in V$, so $s_\alpha^2 = 1$. □

Proposition. If $v, w \in V$ then $(s_\alpha v, s_\alpha w) = (v, w)$.

Proof. This statement generalizes the fact from planar geometry that reflection across a line preserves angles. Algebraically, the result follows by using bilinearity to expand

$$\begin{aligned}(s_\alpha v, s_\alpha w) &= \left(v - \tfrac{2(v,\alpha)}{(\alpha,\alpha)}\alpha, w - \tfrac{2(w,\alpha)}{(\alpha,\alpha)}\alpha\right) \\ &= (v, w) - \tfrac{2(v,\alpha)(w,\alpha)}{(\alpha,\alpha)} - \tfrac{2(v,\alpha)(w,\alpha)}{(\alpha,\alpha)} + \tfrac{4(v,\alpha)(w,\alpha)(\alpha,\alpha)}{(\alpha,\alpha)^2} = (v, w).\end{aligned}$$

□

We write $\mathrm{GL}(V)$ for the *general linear group* of V, consisting of all invertible linear maps $V \to V$. The *orthogonal group* of V with respect to the form $(\cdot,\cdot)$ is the group $O(V)$ consisting of all maps $g \in \mathrm{GL}(V)$ that preserve our bilinear form, i.e., with $(gv, gw) = (v, w)$ for $v, w \in V$. These sets are groups with respect to composition of linear maps.

The preceding proposition amounts to saying that each reflection s_α belongs to $O(V)$.

Definition. A *(finite) reflection group* is a (finite) subgroup of $O(V)$ generated by $\{s_\alpha : \alpha \in X\}$ for some finite set of nonzero vectors $X \subset V \setminus \{0\}$.

The goal of the next few lectures will be to classify the finite reflections groups, that is, to describe which finite groups arise as reflection subgroups of $O(V)$ for some choice of V and the accompanying bilinear form. It turns out, surprisingly, that such a classification is possible and nontrivial. This will afford concrete realizations of many Coxeter groups, and motivate the study of Coxeter systems in general.

Example (Dihedral groups)**.** Let $V = \mathbb{R}^2$ with the standard bilinear form. Fix a regular m-gon centered at the origin. Let D_m be the set of the following linear transformations:

(i) rotation counter-clockwise by angle $\frac{2\pi j}{m}$ for $j = 0, 1, 2, \dots, m-1$,

(ii) reflection across one of the $2m$ "diagonals" of our m-gon (that is, across a line through the origin that either connects two opposite vertices, two midpoints of opposite sides, or a vertex to the midpoint of the opposite side).

There are m distinct transformations of each of these types, so $|D_m| = 2m$. One can check that D_m is a group with respect to composition: this is the group of all rigid motions of $\mathbb{R}^2$ that preserve our regular polygon. Moreover, D_m is a reflection group since rotation by angle $\frac{2\pi}{m}$ is a product of two diagonal reflection. (Try to visualize this for $m = 5$ and $m = 6$.) Call D_m the *dihedral group* of size $2m$ or the *Coxeter group of type* $I_2(m)$.

Example (Symmetric groups)**.** Recall that S_n is the symmetric group of permutations of $[n] = \{1, 2, \dots, n\}$. View S_n as a subgroup of $O(n, \mathbb{R})$, the orthogonal group of $V = \mathbb{R}^n$ with the standard form, by having $w \in S_n$ act on the standard basis $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ via $w(e_i) = e_{w(i)}$, and extending linearly. (Check yourself that this action preserves the standard form $(\cdot,\cdot)$.)

Now $w \in S_n$ corresponds to the matrix $(a_{ij})_{i,j\in[n]}$ with $a_{ij} = 1$ if $j = w(i)$ and 0 otherwise.

Recall that $S_n = \langle s_1, s_2, \dots, s_{n-1}\rangle$ where $s_i = (i, i+1)$ transposes i and $i+1$.

Fact. With respect to our inclusion $S_n \hookrightarrow O(n, \mathbb{R})$, we have $s_i = s_\alpha$ for $\alpha = e_i - e_{i+1}$.

Proof. If $i \in [n-1]$ and $j \in [n]$ then

$$s_{e_i - e_{i+1}}(e_j) = e_j - \tfrac{2(e_i - e_{i+1}, e_j)}{(e_i - e_{i+1}, e_i - e_{i+1})}(e_i - e_{i+1}) = e_j - \delta_{i,j}(e_i - e_{i+1}) + \delta_{i+1,j}(e_i - e_{i+1}).$$

The last expression simplifies to e_i if $j = i+1$, to $e_{i=1}$ if $j = i$, and to e_j otherwise, which is $e_{s_i(j)}$. □

Thus S_n is (isomorphic) to a finite reflection group: this group is the *Coxeter group of type* A_{n-1}.

3 Root systems

To classify reflection groups, we develop some general theory about the action of such a group on the ambient vector space. Continue to let V be a real vector space with a symmetric, positive definite, bilinear form $(\cdot,\cdot)$. Let $W \subset O(V)$ be a finite reflection group.

Proposition. If $w \in O(V)$ and $0 \neq \alpha \in V$ then $w s_\alpha w^{-1} = s_{w\alpha}$. Hence if $w, s_\alpha \in W$ then $s_{w\alpha} \in W$.

Proof. We have $ws_\alpha w^{-1}(w\alpha) = ws_\alpha(\alpha) = w(-\alpha) = -w\alpha = s_{w\alpha}(w\alpha)$. To show that $ws_\alpha w^{-1} = s_{w\alpha}$, it suffices to check that $ws_\alpha w^{-1}(\beta) = \beta = s_{w\alpha}(\beta)$ for all $\beta \in V$ with $(w\alpha, \beta) = 0$. But if $(w\alpha, \beta) = 0$ then

$$0 = (w\alpha, \beta) = (w^{-1}w\alpha, w^{-1}\beta) = (\alpha, w^{-1}\beta),$$

so $s_\alpha(w^{-1}\beta) = w^{-1}\beta$ and therefore $ws_\alpha w^{-1}(\beta) = \beta$. □

When s is a reflection in $O(V)$, let L_s be the line spanned by any nonzero vector $\alpha \in V$ with $s = s_\alpha$. Note that L_s determines s, and that we get the same line for any choice of α.

The proposition shows that W permutes the set of lines $\{L_s : s \text{ is a reflection in } W\}$. To study the structure of W, we should consider this action closely. But rather than work with lines, let's instead replace each line by a pair of opposite vectors and examine W's action on the resulting set of vectors.

This sequences of ideas motivates the following definition of the *root system* of a reflection group.

Definition. Let Φ be a finite set of nonzero vectors in V such that

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for each $\alpha \in \Phi$.

(R2) $s_\alpha(\beta) \in \Phi$ for all $\alpha, \beta \in \Phi$.

Call Φ a *root system*, and refer to its elements as *roots*.

If $W = \langle s_\alpha : \alpha \in \Phi\rangle$, then we say that W is the *reflection group associated to* Φ.

We have the following correspondence between finite reflection groups and root systems.

Proposition. If $W \subset O(V)$ is a finite reflection group then W is the reflection group associated to some root system Φ (though this many not be unique).

Proof. Construct Φ by including the pair of unit vectors on the line L_s for each reflection $s \in W$. This set is finite since W is finite; Φ obviously satisfies (R1); and (R2) holds by the previous proposition. □

Proposition. If Φ is a root system and W is its associated reflection group, then W is finite.

Proof. Let $U = \mathbb{R}\text{-span}\{\alpha \in \Phi\}$ and $U^\perp = \{v \in V : (u, v) = 0 \text{ for all } u \in U\}$. Then for each $\alpha \in \Phi$ we have $s_\alpha u = u$ for all $u \in U^\perp$, so $wu = u$ for all $u \in U^\perp$. We deduce that if $w\alpha = \alpha$ for all $\alpha \in \Phi$ then w fixes all elements of $V = U \oplus U^\perp$, so $w = 1$. Thus the homomorphism $W \to S_n$ for $n = |\Phi|$ induced by the action of W on Φ has trivial kernel so is injective, and so $|W| \leq |S_n| = n! < \infty$. □

Moral: from any finite reflection group we can construct a root system, and the reflections indexed by a root system generate a finite reflection group. So we should develop some theory about root systems.

Definition. A *total order* on V is a transitive relation $<$ such that

(1) $\lambda < \mu$ or $\lambda = \mu$ or $\mu < \lambda$ for each $\lambda, \mu \in V$.

(2) If $\mu < \nu$ then $\lambda + \mu < \lambda + \nu$ for each $\lambda, \mu, \nu \in V$.

(3) If $\lambda < \mu$ then $c\lambda < c\mu$ and $-c\mu < -c\lambda$ for $\lambda, \mu \in V$ and any real number $c > 0$.

This list of conditions looks a little technical, but really just axiomatizes the most natural properties of the usual total order on real numbers.

With respect to a total order $<$ on V, a vector $v \in V$ is *positive* if $0 < v$. Positive vectors are preserved under sums and under products by positive scalars. An important thing to note:

Proposition. A total order $<$ exists on V.

Proof. Let $e_1, e_2, \ldots, e_n$ be an arbitrary basis of V. Set $\lambda < \mu$ if we have $\lambda = a_1e_1 + a_2e_2 + \cdots + a_ne_n$ and $\mu = b_1e_1 + b_2e_2 + \ldots b_ne_n$, where each $a_i, b_i \in \mathbb{R}$, and it holds that $a_j < b_j$ for some $j \in [n]$ while $a_i = b_i$ for $1 \leq i < j$. Check that this relation is transitive and satisfies the axioms of a total order. □

We refer to the total order constructed in the preceding proof and the *lexicographic order* induced by the ordered basis $e_1, e_2, \ldots, e_n$.

There are several important constructions attached to a root system which depend on a choice of total order on V. The relevant definitions can seem a little unnatural, since at our current level of abstraction there is no obviously "best" total order to adopt. We will see, however, that all useful definitions depending on a choice of total order are actually independent of our choice.

Definition. Let Φ be a root system. A subset $\Pi \subset \Phi$ is a *positive system* if every $\alpha \in \Phi$ is positive (i.e., $0 < \alpha$) with respect to some total order $<$ on V.

Proposition. If $\Pi \subset \Phi$ is a positive system then $\Phi = \Pi \sqcup -\Pi$ where $\sqcup$ denotes disjoint union.

Proof. This follows since roots in Φ come in pairs $\{-\alpha, \alpha\}$. □

Definition. A subset $\Delta \subset \Phi$ is a *simple system* if Δ is a linearly independent set of vectors and each $\alpha \in \Phi$ can be expressed as $\alpha = \sum_{\beta\in\Delta} c_\beta \beta$ for coefficients $c_\beta \in \mathbb{R}$ which are either all ≥ 0 or all ≤ 0. Elements of a simple system are called *simple roots*.

It is not obvious that every root system contains a simple system. (Why is it obvious that every root system contains a positive system?) Nevertheless, the following is true:

Theorem. Let Φ be a root system.

(a) If Δ is a simple system in Φ, then there is a unique positive system $\Pi \subset \Phi$ containing Δ.

(b) Every positive system $\Pi \subset \Phi$ contains a unique simple system. Thus, simple systems always exist.

Proof. Today, we prove the first part and start the second.

(a) The unique positive system Π containing a given simple system Δ is the one defined with respect to the lexicographic total order induced by any ordering of Δ. This positive system is uniquely characterized as the intersection $\Phi \cap \mathbb{R}^+\text{-span}\{\alpha \in \Delta\}$ where $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$.

(b) Suppose Δ is a simple system contained in a positive system Π. Then Δ is the unique simple system in Π since Δ is the subset of roots $\lambda \in \Pi$ such that $\lambda \neq \alpha + \beta$ for all $\alpha, \beta \in \Pi$.

    It remains to construct a simple system $\Delta \subset \Pi$. The idea is to let Δ be the minimal subset of Π such that each $\alpha \in \Pi$ is a nonnegative linear combination of elements of Δ. This set will be a simple system if we can show that it is linearly independent—which we will do next time!

□