1 Last time: reflection groups and root systems

Recall our usual setup: let V be a vector space over the real numbers $\mathbb{R}$, with a symmetric, positive definite, bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$.

The *reflection* with respect to a nonzero vector $\alpha \in V$ the linear map

$$s_\alpha : v \mapsto v - \tfrac{2(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

The terminology comes from the fact that the vector $s_\alpha v$ is given by "reflecting" v across the hyperplane orthogonal to α.

A *reflection group* is a subgroup of the general linear group $\mathrm{GL}(V)$ generated by a finite set of reflections.

Our goal is to classify the finite groups which are reflection groups. Examples of such groups include the dihedral groups, symmetric groups, etc.

Let W be a finite reflection group.

We saw last time that if $s_\alpha \in W$ for some nonzero $\alpha \in V$ then $s_{w\alpha} = ws_\alpha w^{-1} \in W$ for all $w \in W$. Thus W acts on the set of lines spanned by vectors α with $s_\alpha \in W$. This set is finite since W is finite. The notion of a root system gives an abstract model of this action.

A *root system* is a finite set Φ of nonzero vectors in V such that

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for each $\alpha \in \Phi$.

(R2) $s_\alpha(\beta) \in \Phi$ for all $\alpha, \beta \in \Phi$.

Elements of Φ are called *roots*. The group $W = \langle s_\alpha : \alpha \in \Phi\rangle$ is the *reflection group associated to* Φ.

We saw last time that any finite reflection group arises as the group associated to some root system, and that conversely the reflection group associated to any root system is finite.

A *total order* on V is a transitive relation $<$ on V such that

(1) Exactly one of $a < b$ or $a = b$ or $b < a$ holds for each $a, b \in V$.

(2) If $a < b$ then $a + c < b + c$ for all $a, b, c \in V$.

(3) If $x < y$ and $c \in \mathbb{R}$ is positive then $cx < cy$ and $-cy < -cx$.

This list of conditions looks long, but just encodes our usual intuitions about total orderings of numbers.

The easiest way to construct a total order on V is to choose a basis $v_1, v_2, \dots, v_n$ and set $\sum_i a_i v_i < \sum_i b_i v_i$ if for some $j \in [n]$ it holds that $a_1 = b_1$, $a_2 = b_2$, …, $a_{j-1} = b_{j-1}$, and $a_j < b_j$.

Fact. Let $<$ be a total order on V. If $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in V$ are such that $a_i < b_i$ for all $i \in [n]$, then $\sum_i a_i < \sum_i b_i$.

Proof. This follows by induction: if $\sum_{i=1}^{n-1} a_i < \sum_{i=1}^{n-1} b_i$ then $\sum_{i=1}^{n} a_i < a_n + \sum_{i=1}^{n-1} b_i < b_n + \sum_{i=1}^{n-1} b_i$. □

A *positive system* Π in a root system Φ is a set of the form $\{\alpha \in \Phi : 0 < \alpha\}$ where $<$ is some total order on V. Once we have chosen a total order, there is only one corresponding positive system in Φ, but if we haven't specified $<$ then there are many choices for Π. For example, $-\Pi$ is also a positive system (since "$>$" is also a total order). The set Φ is the disjoint union of Π and $-\Pi$.

A *simple system* Δ in a root system Φ is a subset of linearly independent roots with the property that every $\alpha \in \Phi$ can be written (uniquely) as $\alpha = \sum_{\beta\in\Delta} c_\beta \beta$ where either $0 \le \min_{\beta\in\Delta} c_\beta$ (all coefficients positive) or $\max_{\beta\in\Delta} c_\beta \le 0$ (all coefficients negative).

Positive systems obviously exist (why?), but it is not immediate from the definition that every root system has a simple system. So the following result is nontrivial:

Theorem. Every simple system in a root system is contained in a unique positive system. Every positive system in a root system contains a unique simple system.

Proof. We proved the first statement last time. To prove the second, suppose Π is a positive system in a root system Φ. Let Δ be the minimal subset of Π such that each $\alpha \in \Pi$ is a nonnegative linear combination of elements of Δ. To show that Δ is a simple system, it is enough to check that the set is linearly independent. For this, we need the following lemma:

Lemma. Let $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Then $(\alpha, \beta) \leq 0$.

Proof. Suppose $(\alpha, \beta) > 0$. We argue by contradiction. Note that

$$s_\alpha \beta = \beta - c\alpha \tag{*}$$

for $c = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} > 0$. Suppose first that $s_\alpha\beta \in \Pi$. We then have

$$s_\alpha \beta = \sum_{\gamma \in \Delta} c_\gamma \gamma \tag{**}$$

for some nonnegative coefficients c_γ, by definition. If the coefficient $c_\beta < 0$ then comparing (*) and (**) shows that $(1 - c_\beta)\beta$ is a nonnegative linear combination of the elements of $\Delta \setminus \{\beta\}$, so the same is true of β since $1 - c_\beta > 0$; but this means we cannot have $\beta \in \Delta$, contradicting our assumption otherwise. If $c_\beta \geq 1$ then a similar comparison shows that $0 = (c_\beta - 1)\beta + c\alpha + \sum_{\gamma \in \Delta \setminus \{\beta\}} c_\gamma \gamma$. This is a contradiction since the fact proved above implies $0 < (c_\beta - 1)\beta + c\alpha + \sum_{\gamma\in\Delta\setminus\{\beta\}} c_\gamma\gamma$.

A similar argument leads to a contradiction if we assume instead that $s_\alpha\beta \in -\Pi$. Since $s_\alpha\beta$ must belong to Π or $-\Pi$ since the union of these two is Φ, we relent and deduce that $(\alpha, \beta) \leq 0$. □

The proof of the theorem from the lemma goes as follows: if Δ were not linearly independent then we could write $\sum b_\beta \beta = \sum c_\gamma \gamma \neq 0$ where the sums are over disjoint subsets of Δ and the coefficients b_β and c_γ are > 0. Let σ denote the common value of these linear combinations. We would then have $0 \leq (\sigma, \sigma) = \sum_{\beta,\gamma} b_\beta c_\gamma (\beta, \gamma) \leq 0$, the first $\leq$ by positive definiteness and the second by the lemma. This implies that $(\sigma, \sigma) = 0$ so $\sigma = 0$, which is a contradiction.

Conclude that Δ is a simple system. (Why is it unique?) □

Let us restate the lemma in the theorem as a corollary:

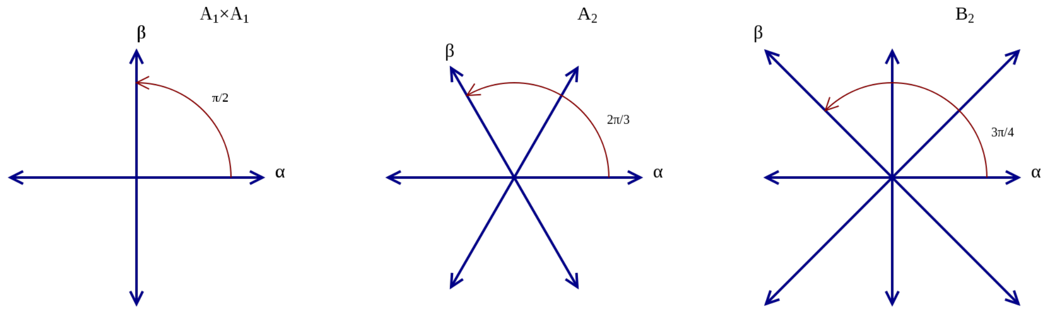
Corollary. If Δ is a simple system in Φ then $(\alpha, \beta) \leq 0$ for all distinct $\alpha, \beta \in \Delta$.

We sometimes refer to the size of any simple system $\Delta \subset \Phi$ as the *rank* of the group $W = \langle s_\alpha : \alpha \in \Phi \rangle$.

2 Examples of root systems

What we have shown so far: given a root system Φ, we can always choose a total order $<$, which determines a positive system Π, which determines a simple system Δ. There is a lot to unpack in this statement, so we digress briefly with some simple examples of root systems.

If $V = \mathbb{R}^2$ with the standard bilinear form $(v, w) = v_1 w_1 + v_2 w_2$, then the following sets of 4, 6, and 8 vectors in V are root systems (namely, of *types* $A_1 \times A_1$, A_2, and B_2):



The vectors labeled α and β in each picture make up simple system. (Why can there only be two simple roots if $V = \mathbb{R}^2$? What is the corresponding positive system and total order on V?)

We should also mention an example that lives in higher dimensions. Let V be the subset of vectors $v \in \mathbb{R}^n$ whose coefficients in the standard basis sum to zero, i.e., with $\sum_{i=1}^n v_i = 0$. This is a subspace of dimension $n-1$. Take $(\cdot, \cdot)$ to be the usual bilinear form on $\mathbb{R}^n$ restricted to V.

Define $<$ as the total order on $\mathbb{R}^n$ induced by lexicographic order on the standard basis $e_1, e_2, \dots, e_n$.

Define $\Phi = \{e_i - e_j : 1 \leq i, j \leq n \text{ and } i \neq j\}$. For example, if $n = 3$ then

$$\Phi = \{e_1 - e_2, e_1 - e_3, e_2 - e_3, e_2 - e_1, e_3 - e_1, e_3 - e_2\}.$$

Exercise:

(1) Φ is a root system.

(2) $\Pi = \{e_i - e_j : 1 \leq i < j \leq n\}$ is a positive system in Φ.

(3) $\Delta = \{e_i - e_{i+1} : i \in [n-1]\}$ is a simple system in Π.

Note at least that if $i \neq j$ then $(e_i - e_{i+1}, e_j - e_{j+1})$ is either 0 or -1, as our earlier lemma predicted.

A natural thing to inquire: what is the associated reflection group $W = \langle s_\alpha : \alpha \in \Phi \rangle$?

Observe that

$$s_{e_i - e_j}(v) = v - \tfrac{2(v, e_i - e_j)}{(e_i - e_j, e_i - e_j)}(e_i - e_j) = v - (v_i - v_j)(e_i - e_j) = (v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

Thus $s_{e_i - e_j}$ acts on V by transposing the ith and jth coordinates of vectors. It follows that there exists an isomorphism $S_n \to W$ mapping the transposition (i, j) to $s_{e_i - e_j}$.

(What are the reflection groups associated to the example root systems in $\mathbb{R}^2$?)

3 Relating different simple systems

Let V be a real vector space with the usual symmetric, positive definite, bilinear form $(\cdot, \cdot)$.

Let $\Phi \subset V$ be a root system, $\Pi \subset \Phi$ a positive system, and $\Delta \subset \Pi$ a simple system. There are a lot of implicit choices made here. How important are these choices? As our last results today we will show that the answer to this question is: not very.

Let $W = \langle s_\alpha : \alpha \in \Phi \rangle$.

Proposition. If $\alpha \in \Delta$ then $s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}$.

Proof. Let $\beta \in \Pi \setminus \{\alpha\}$. Write $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$, where each coefficient $c_\gamma \in \mathbb{R}$ is ≥ 0. Since $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$, there exists $\gamma \neq \alpha$ in Δ with $c_\gamma > 0$. Hence

$$s_\alpha \beta = \beta - c\alpha = \sum_{\gamma \in \Delta \setminus \{\alpha\}} c_\gamma \gamma + (c_\alpha - c)\alpha$$

for $c = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. The last expression is a root, so it must be a positive root, since the coefficient c_γ is positive. For the same reason, this positive root cannot be α. Therefore $s_\alpha \beta \in \Pi \setminus \{\alpha\}$, so s_α maps $\Pi \setminus \{\alpha\} \to \Pi \setminus \{\alpha\}$. Since s_α is invertible, this map is a bijection. □

Remember that $s_\alpha \alpha = -\alpha$.

Corollary. Let $\alpha \in \Delta$. Then $\{\beta \in \Pi : s_\alpha \beta \in -\Pi\} = \{\alpha\}$.

The group W not only preserves Φ, but permutes the set of simple/positive systems in Φ:

Theorem. Any two positive (respectively, simple) systems in Φ are conjugate under W, in the sense that if Π, Π' are positive systems and Δ, Δ' are the unique simple systems they contain, then there exists $w \in W$ with $w\Pi = \Pi'$ and $w\Delta = \Delta'$.

Proof. It suffices to show this for positive systems. (Why?) Let Π and Π' be two positive systems in Φ Note that $|\Pi| = |\Pi'| = |\Phi|/2$. We argue by induction on the number $r = |\Pi \cap -\Pi'|$. If $r = 0$ then $\Pi = \Pi'$, so take $w = 1$. Assume $r > 0$, so that $\Pi \not\subset \Pi'$. Let Δ be the unique simple system in Π. Then certainly also $\Delta \not\subset \Pi'$. Choose $\alpha \in \Delta$ with $\alpha \in -\Pi'$. Then $|s_\alpha \Pi \cap -\Pi'| = r - 1$ by the proposition. But $s_\alpha \Pi$ is also a positive system (why?) so by induction there exists $w \in W$ with $w(s_\alpha \Pi) = \Pi'$, i.e., $(ws_\alpha)\Pi = \Pi'$. □

Our final theorem today is an analogue of a result we saw in the first lecture for the symmetric group.

Define a *simple reflection* to be a reflection s_α where $\alpha \in \Delta$ is a simple root. Define the *height* of $\beta \in \Phi$ relative to Δ as $\mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha$ where $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$.

Theorem. Let $W = \langle s_\alpha : \alpha \in \Phi \rangle$. Then $W = \langle s_\alpha : \alpha \in \Delta \rangle$.

Proof. Let $W' = \langle s_\alpha : \alpha \in \Delta \rangle$. Clearly $W' \subset W$. We want to show that $W \subset W'$. We will deduce this from two claims.

Claim. If $\beta \in \Pi$ and γ is an element of minimal height in $W'\beta \cap \Pi$ then $\gamma \in \Delta$.

Proof of claim. Write $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Note that every $c_\alpha \geq 0$ but $0 < (\gamma,\gamma) = \sum_\alpha c_\alpha(\gamma,\alpha)$, so $(\gamma,\alpha) > 0$ for some $\alpha \in \Delta$. If $\gamma \neq \alpha$ then $s_\alpha \gamma$ is positive by the previous proposition, but $\mathrm{ht}(s_\alpha \gamma) = \mathrm{ht}(\gamma) - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)} < \mathrm{ht}(\gamma)$ and $s_\alpha \gamma \in W'\beta$ since $s_\alpha \in W'$, contradicting the minimality of the height of γ. So $\gamma = \alpha \in \Delta$. □

Claim. $W'\Delta = \Phi$.

Proof of claim. $\Pi \subset W'\Delta$ since the W'-orbit of each $\beta \in \Pi$ intersects Δ by the first claim. If $\beta \in -\Pi$ then $-\beta = w\alpha$ for some $w \in W'$ and $\alpha \in \Delta$, so $\beta = ws_\alpha \alpha \in W'\Delta$, since $ws_\alpha \in W'$ and $s_\alpha \alpha = -\alpha$. □

To prove the theorem using the second claim, note that if s_β is a generator of W for some $\beta \in \Phi$ then $\beta = w\alpha$ for some $w \in W'$ and $\alpha \in \Delta$ (by the claim), so $s_\beta = ws_\alpha w^{-1} \in W'$. This means $\{s_\beta : \beta \in \Phi\} \subset W'$ so $W \subset W'$ as we wanted to show! □

We can restate the second claim as the following, now that we know that $W = W'$.

Corollary. If $\beta \in \Phi$ then there exists $w \in W$ with $w\beta \in \Delta$.