1 Last time: length functions and the exchange principle

Recall our familiar setup:

1. V is a real vector space with a symmetric, positive definite, bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$.
2. $\Phi \subset V$ is a root system.
3. $\Pi \subset \Phi$ is a positive system. We write $\alpha > 0$ to mean $\alpha \in \Pi$.
4. $\Delta \subset \Pi$ is a simple system.

Let $S = \{s_\alpha : \alpha \in \Delta\}$ and $W = \langle s_\alpha : \alpha \in \Phi \rangle$

Nontrivial facts: W is finite and generated by just S, and $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ in Δ.

From last time: the *length* of $w \in W$ (with respect to the choices of Δ, Π, and Φ) is the smallest integer $r \geq 0$ such that $w = s_1 s_2 \cdots s_r$ where each $s_i \in S$. Denote this length as $\ell(w)$.

Proposition. If $w \in W$ then $\ell(w) = n(w)$, where $n(w)$ is the number of $\alpha \in \Pi$ with $w\alpha \notin \Pi$.

Theorem (Exchange principle)**.** Let $w = s_1 s_2 \cdots s_r$ where each $s_i \in S$. If $\ell(w) < r$ then there exist indices $1 \leq i < j \leq r$ such that

(a) $s_{i+1} s_{i+2} \cdots s_j = s_i s_{i+1} \cdots s_{j-1}$.

(b) $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$ where for each $\widehat{\ }$ we omit the capped factor.

Say that $w = s_1 s_2 \cdots s_r$ is a *reduced expression* for $w \in W$ if $s_i \in S$ and $r = \ell(w)$.

Corollary. If $w = s_1 s_2 \cdots s_r$ is not a reduced expression for w, then we can obtain one by deleting an even number of factors.

Corollary. Let $w = s_1 s_2 \cdots s_r$ where $s_i \in S$, and let $s \in S$.

(a) If $\ell(ws) < \ell(w)$ then there is an index $i \in \{1, 2, \ldots, r\}$ with $ws = s_1 \cdots \widehat{s_i} \cdots s_r$,

(b) If $\ell(w) = r$ then this index is unique.

Thus $\ell(ws) < \ell(w)$ if and only if w has a reduced expression ending in s.

Proof. Assume $r = \ell(w) > \ell(ws)$ and consider the expression $ws = s_1 \cdots s_r s$. Since this is not reduced, we can omit two factors without changing the product. If neither of this factors is the right-most factor s, then by canceling this factor we could obtain an expression for w with $r-2$ factors, contradicting the fact that $\ell(w) = r$. So one of the omitted factors is s, meaning that we have $ws = s_1 \cdots \widehat{s_i} \cdots s_r$ for some i. If there were another index j with $ws = s_1 \cdots \widehat{s_j} \cdots s_r$ then it would follow that $ws = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$, contradicting the fact that $\ell(ws) = r - 1$.

The argument to deduce the result when $\ell(w) < r$ is left as an exercise. □

2 The longest element

Summarizing some of the properties we have recently seen:

Theorem. The following are equivalent for $w \in W$:

(a) $w = 1$.

(b) $w\Pi = \Pi$.

(c) $w\Delta = \Delta$.

(d) $n(w) = 0$

(e) $\ell(w) = 0$.

We also proved in an earlier lecture:

Theorem. W acts transitively on the set of positive (respectively, simple) systems in Φ.

Corollary. For two positive (simple) systems Π, Π' (Δ, Δ') in Φ, there exists a unique $w \in W$ with $w\Pi = \Pi'$ $(w\Delta = \Delta')$.

If Π is a positive system, then by symmetry, $-\Pi$ is also a positive system (with respect to the opposite total order) in Φ. Combining this observation with the preceding theorems gives:

Corollary. There exists a unique element $w_0 \in W$ with $w\Pi = -\Pi$. This is the *longest element* of W, since it is the unique element with $\ell(w_0) = |\Pi| = \frac{|\Phi|}{2}$.

Proposition. If holds that $\ell(sw_0) = \ell(w_0 s) = \ell(w_0) - 1$ for all $s \in S$.

Proof. Otherwise the length would increase, which is impossible. □

Proposition. If $v \in W$ then $\ell(w_0 v) = \ell(w_0) - \ell(v)$.

Proof. Keep multiplying v on the right by simple reflections increasing the length, while this is possible. The end result will be an element $w \in W$ with $w\alpha \in -\Pi$ for all $\alpha \in \Delta$, so $w\alpha \in -\Pi$ for all $\alpha \in \Pi$. Therefore $w = w_0$, and we can write $vu = w_0$ where u has length $\ell(w_0) - \ell(v)$. and it follows that $\ell(w_0 v) = \ell(v^{-1} w_0) = \ell(u) = \ell(w_0) - \ell(v)$. □

3 Presentations and Coxeter systems

Here is the main theorem of today, promised last time:

Theorem. Define $m(s,t)$ for $s, t \in S$ as the order of the product st in W, that is, the smallest integer $n \geq 1$ with $(st)^n = 1$. The reflection group W then has the presentation

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle.$$

There always exists a surjective homomorphism from the group on the left to W; the nontrivial part of the theorem is the claim that this homomorphism is injective.

A consequence of this result is that if G is a group and $f : S \to G$ is any map, then f extends to a group homomorphism $W \to G$ if and only if $(f(s)f(t))^{m(s,t)} = 1$ for all $s, t \in S$.

Since $s^2 = 1$ for $s \in S$, the relation $(st)^{m(s,t)} = 1$ is equivalent to

$$\underbrace{ststststs\cdots}_{m(s,t)\text{ factors}} = \underbrace{tstststst\cdots}_{m(s,t)\text{ factors}}.$$

We call this a *braid relation* for W.

Proof. We argue informally that any relation

$$s_1 \cdots s_r = 1 \tag{1}$$

which holds in W (where $s_i \in S$) can be deduced from the braid relations. (This amounts to showing that the kernel of the natural homomorphism from our finitely presented group to W is trivial.)

Note that r must be even in (1) since $\det s_i = -1$. If $r = 2$, then $s_1 s_2 = 1$ implies that $s_1 = s_2^{-1} = s_2$, so (1) is the given relation $s_1 s_1 = 1$.

We proceed by induction on $r = 2q$, with $q > 1$. Throughout, we make use of the fact that we can always cancel any factors we want to rewrite various expression, since such cancellations follow from the relations $s^2 = 1$. For example, (1) implies that

$$s_1 \cdots s_{q+1} = s_r \cdots s_{q+2}.$$

The left side cannot be reduced so by the exchange condition there are indices $1 \leq i < j \leq q+1$ such that $s_{i+1} \cdots s_j = s_i \cdots s_{j-1}$ holds in W, giving

$$s_i s_{i+1} \cdots s_{j-1} s_j s_{j-1} \cdots s_{i+1} = 1. \tag{2}$$

If the left side of this relation has fewer than r factors, then we may assume that the relation is implied by the braid relations, and obtain by induction that

$$s_1 \cdots s_r = s_1 \cdots s_i (s_i \cdots s_{j-1}) s_{j+1} \cdots s_r = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r = 1$$

is implied by the braid relations.

This conclusion fails only if (2) has r factors, which holds if $i = 1$ and $j = q+1$ in which case (2) is

$$s_2 \cdots s_{q+1} = s_1 \cdots s_q. \tag{3}$$

Suppose we instead apply the above steps to try to deduce the relation

$$s_2 \cdots s_r s_1 = 1$$

which is equivalent to (1), from the braid relations By the same argument, we will be successful unless

$$s_3 \cdots s_{q+2} = s_2 \cdots s_{q+1}.$$

Rewrite this last relation as

$$s_3 (s_2 s_3 \cdots s_{q+1})(s_{q+2} s_{q+1} \cdots s_r) = 1.$$

Applying the same argument again will success unless

$$s_2 s_3 \cdots s_{q+1} = s_3 s_2 s_3 \cdots s_q.$$

Substituting (3) into this equation and canceling factors then gives $s_1 = s_3$.

Applying this technique to all of the equivalent relations

$$s_1 \cdots s_r = 1$$
$$s_2 \cdots s_r s_1 = 1$$
$$s_3 \cdots s_r s_1 s_2 = 1$$

and so forth, we deduce that either we can generate one (therefore all) of these relations from braids, or

$$s_1 = s_3 = s_5 = \cdots = s_{r-1} \qquad \text{and} \qquad s_2 = s_4 = s_6 = \cdots = s_r$$

in which case (1) holds automatically since it is necessarily one of the given relations. □

This result at last brings us to the definition of a *Coxeter system*.

Definition. A *Coxeter system* is a pair (W, S) where W is a group and $S \subset W$ is a set of elements of order two which generate W, such that

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for all } s, t \in S \rangle$$

where $m(s,t)$ denotes the order of $st \in W$.

(This order may be infinite, in which case the relation $(st)^\infty = 1$ is ignored in the presentation.)

A *Coxeter group* is a group which occurs as the group W in some Coxeter system.

Corollary. Each finite reflection group is a Coxeter group.