1 Last time: reflection groups as Coxeter groups

Recall our familiar setup:

1. V is a real vector space with a symmetric, positive definite, bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$.
2. $\Phi \subset V$ is a root system.
3. $\Pi \subset \Phi$ is a positive system. We write $\alpha > 0$ to mean $\alpha \in \Pi$.
4. $\Delta \subset \Pi$ is a simple system.

Let $S = \{s_\alpha : \alpha \in \Delta\}$ and $W = \langle s_\alpha : \alpha \in \Phi\rangle$

The *length* of $w \in W$ (with respect to the choices of Δ, Π, and Φ) is the smallest integer $r \geq 0$ such that $w = s_1 s_2 \cdots s_r$ where each $s_i \in S$. Denote this length as $\ell(w)$. This is also the number of positive roots $\alpha \in \Pi$ such that $w\alpha$ is not positive.

The *exchange principle* is the fact that if $w = s_1 s_2 \cdots s_r$ where each $s_i \in S$ and $\ell(w) < r$, then we can delete exactly two factors s_i and s_j without changing the product.

Define $m(s,t)$ for $s,t \in S$ as the smallest integer $n > 0$ such that $(st)^n = 1$, i.e., the order of st in W.

Theorem. The reflection group W then has the presentation

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for all } s,t \in S\rangle.$$

Therefore, if G is a group and $f : S \to G$ is any map, then f extends to a group homomorphism $W \to G$ if and only if $(f(s)f(t))^{m(s,t)} = 1$ for all $s,t \in S$.

The theorem shows that every finite reflection group is a *Coxeter group*, which is defined as follows:

Definition. A *Coxeter group* is a group W generated by a set S with a presentation of the form

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s,t \in S \text{ with } m(s,t) \neq \infty\rangle$$

where $m : S \times S \to \{1,2,3,\dots\} \cup \{\infty\}$ is a map with $m(s,t) = m(t,s)$ and $m(s,s) = 1$ for all $s,t \in S$.

The pair (W,S) is a *Coxeter system*.

2 Coxeter graphs

A *graph* $\mathcal{G} = (V,E)$ is a pair consisting of a set of vertices V, and a set E of subsets of V of size two, which we call edges. A *weighted graph* is a graph $\mathcal{G} = (V,E)$ plus a map which assigns a weight (e.g., a number, a color, etc.) to each edge. All of graphs today will be weighted graphs with edge weights given by elements of $\{3,4,5,\dots\} \cup \{\infty\}$.

A Coxeter system (W,S) may be encoded by its *Coxeter graph* (which we'll also sometimes call the *Coxeter diagram*) which is defined as the weighted graph with vertex set S and with an edge labeled by $m(s,t)$ from s to t whenever $s,t \in S$ are such that $m(s,t) > 2$.

For example:

$$a \overset{3}{\text{---}} b \overset{4}{\text{---}} c$$

is the Coxeter graph of $W = \langle a,b,c : a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ac)^2 = 1\rangle$. By convention, people often omit any edge weights equal to 3 from these diagrams. An example using this convention:

$$\begin{array}{ccc} a & \text{---} & b \\ | & & | \\ c & \text{---} & d \end{array}$$

is the graph of $W = \langle a,b,c,d : a^2 = b^2 = c^2 = d^2 = (ad)^2 = (bc)^2 = (ab)^3 = (bd)^3 = (cd)^3 = (ac)^3\rangle$.

Example (Symmetric groups)**.** Let $\Phi = \{e_i - e_j : i \neq j\} \subset \mathbb{R}^n$ and $\Delta = \{e_i - e_{i+1} : 1 \leq i < n\}$.

Recall that $W = \langle s_1, s_2, \dots, s_{n-1}\rangle \cong S_n$ for $s_i = s_{e_i - e_{i+1}}$ since $s_i \mapsto (i, i+1)$ extends to an isomorphism.

Let $S = \{s_1, s_2, \dots, s_{n-1}\rangle$. For $s_i, s_j \in S$, we then have:

- $m(s_i, s_j) =$ order of 1 if $i = j$.
- $m(s_i, s_j) =$ order of $(i, i+1, i+2)$ or $(i, i+2, i+1)$ if $|i - j| = 1$.
- $m(s_i, s_j) =$ order of $(i, i+1)(j, j+1)$ if $|i - j| > 1$.

Thus $m(s_i, s_j)$ is 1, 2 or 3 according to whether $|i - j|$ is 0, 1, or ≥ 2.

It follows that the Coxeter graph of (W, S) is

$$s_1 — s_2 — \dots — s_{n-1}.$$

Remember that the unlabeled edges are secretly labeled by 3's.

Example (Dihedral groups)**.** Fix $m \geq 3$.

Let $\Phi = \left\{ \begin{bmatrix} \cos(i\theta) \\ \sin(i\theta) \end{bmatrix} : i = 0, 1, \dots, 2m \right\} \subset \mathbb{R}^2$ where $\theta = \pi/m$.

This is a root system. For positive roots, we can take the vectors $\begin{bmatrix} x \\ y \end{bmatrix} \in \Phi$ with $x > 0$ or $x = 0$, $y > 0$.

The corresponding simple system $\Delta = \{\alpha, \beta\}$ then must consist of the two positive roots making the largest angle, since the $\mathbb{R}^+$-span of any two roots stays inside the cone they determine.

The group $W = \langle s_\alpha, s_\beta\rangle$ is a finite reflection group of order $2m$, so it has a presentation of the form

$$W = \langle s_\alpha, s_\beta : s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^? = 1\rangle.$$

The question mark ? stands in for the order of $s_\alpha s_\beta$. To compute this, either note that $s_\alpha s_\beta$ is a rotation by angle $2\pi/m$, or try listing the distinct elements of W and note that $|W| = 2m$.

In this case, the Coxeter graph of (W, S) is

$$s_\alpha \overset{m}{—} s_\beta.$$

If we set $m = \infty$ in this diagram then the corresponding Coxeter group W is the *infinite dihedral group*.

Any graph with edges labeled by elements of $\{3, 4, 5, \dots\} \cup \{\infty\}$ is the graph of a Coxeter system.

Remark. Any (weighted graph) automorphism of a Coxeter diagram extends to an automorphism of the corresponding Coxeter group. (An automorphism of a graph is a permutation of its vertices which preserves edges and edge weights.) This is because any graph automorphism provides us with a map $S \to S$ which preserves all the required relations.

For example, the graph automorphism

$$s_1 — s_2 — \dots — s_{n-1} \qquad \mapsto \qquad s_n — s_{n-1} — \dots — s_1$$

corresponds to the inner automorphism of S_n given by $\sigma \mapsto w_0 \sigma w_0$ for $w_0 = n \cdots 321$.

3 Parabolic subgroups

When the Coxeter graph of a finite reflection group is not connected, each of its connected components is the Coxeter graph of a smaller Coxeter group. As one might hope, each of these Coxeter groups is itself a reflection group. To be precise:

Proposition. Fix a simple system Δ in a root system Φ in a real vector space V (with a bilinear form with the usual properties). Let $S = \{s_\alpha : \alpha \in \Delta\}$ and suppose $J \subset S$. Define

$$\Delta_J = \{\alpha \in \Delta : s_\alpha \in J\} \qquad \text{and} \qquad V_J = \mathbb{R}\text{-span}\{\alpha \in \Delta_J\} \qquad \text{and} \qquad \Phi_J = \Phi \cap V_J.$$

(a) $\Phi_J \subset V$ is a root system with simple system Δ_J and corresponding reflection group $W_J = \langle s \in J \rangle$.

(b) The length function of W_J relative to Δ_J is the restriction of the length function of W.

(c) Define $W^J = \{w \in W : \ell(ws) > \ell(w) \text{ for all } s \in J\}$.

For each $w \in W$, there are unique elements $u \in W^J$ and $v \in W_J$ such that $w = uv$.

It moreover holds that $\ell(w) = \ell(u) + \ell(v)$.

Proof. Part (a) is evident from the axioms of a root system.

For part (b), let Φ^+ be the positive system in Φ containing Δ, and suppose $\alpha \in \Phi^+ \setminus \Phi_J$. Then α involves some simple root $\gamma \notin \Delta_J$, and it follows that $s_\beta\alpha$ still involves γ with a positive coefficient for all $\beta \in \Delta_J$. We deduce that $s_\beta\alpha > 0$ for $\beta \in \Delta_J$, so $w\alpha > 0$ for $w \in W_J$. Thus the roots in Φ^+ sent by $w \in W_J$ to negative roots are precisely the roots in Φ^+_J sent by w to negative roots. It follows that the length function of W_J coincides with the restriction of the length function of W.

For part (c), fix $w \in W$ and choose an element $u \in wW_J$ of smallest possible length. Define $v \in W_J$ so that $w = uv$. We must have $u \in W^J$ since $us \in wW_J$ for all $s \in J$. To show that $\ell(w) = \ell(u) + \ell(v)$ we use the deletion condition. Suppose $u = s_1 \cdots s_q$ and $v = t_1 \cdots t_r$ are reduced expressions. If $\ell(w) < q + r$ then we can omit two factors from $s_1 \cdots s_q t_1 \cdots t_r$ and still get w. Omitting any s_i would lead to a shorter coset representative in wW_J, while omitting two factors from $t_1 \cdots t_r$ would contradict $\ell(v) = r$. Conclude that $\ell(w) = \ell(u) + \ell(v)$.

Our argument applies to any $w \in W$, so it must hold that u is the unique shortest element in wW_J. It remains to show that $W^J \cap wW_J = \{u\}$. Suppose $u' \in W^J \cap wW_J$. Then $u' = uv'$ for some $v' \in W_J$. If $v' \neq 1$ then for any $s \in J$ with $\ell(v's) < \ell(v')$ we would have $\ell(u's = \ell(uv's) = \ell(u) + \ell(v's) < \ell(u')$, a contradiction. Hence $v' = 1$ so $u = u'$. □

Elements of the set W^J are *minimal (left) coset representatives.*

We refer to W_J as a *(standard) parabolic subgroup.*

The proposition shows that (W_J, J) is a Coxeter system.

A Coxeter system (W, S) is *irreducible* if its Coxeter graph is connected, meaning that there is a path following the edges of the graph between any two vertices. To classify the finite reflection groups, it suffices to classify the irreducible ones:

Proposition. Let (W, S) be a finite reflection group with Coxeter graph Γ. Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of Γ, and let S_i be the set of vertices in Γ_i.

Then $W = W_{S_1} \times \cdots \times W_{S_r}$ and each (W_{S_i}, S_i) is irreducible.

Proof. We proceed by induction on r. Since the elements of S_i commute with the elements of S_j for $i \neq j$, it is clear that the indicated parabolic subgroups centralize each other, so each is normal.

The product of these groups contains S so is all of W. By induction $W_{S \setminus S_r} = W_{S_1} \times \cdots \times W_{S_{r-1}}$. It remains to show the indicated subgroups are disjoint. This will derive from the following lemma:

Lemma. If $I, J \subset S$ and $I \cap J = \varnothing$, then $W_I \cap W_J = \{1\}$.

Proof. If $w \in W_I$ then $wu = u$ for all $u \in V_I^\perp$ where

$$V_I^\perp = \{u \in V : (u, v) = 0 \text{ for all } v \in V_I\},$$

since this clearly holds when $w = s_\alpha \in I$. By symmetry if $w \in W_I \cap W_J$ then $wu = u$ for all $u \in V_I^\perp + V_J^\perp$. But it is an exercise in linear algebra to check that

$$V_I^\perp + V_J^\perp = (V_I \cap V_J)^\perp = (V_{I\cap J})^\perp = 0^\perp = V,$$

so if $w \in W_I \cap W_J$ then $w = 1$. □

□

Next time we will sketch the classification of finite reflection groups, and then start the general theory of Coxeter systems.