1   Last time: Coxeter graphs and parabolic subgroups

To start, recall our new definition of principal interest:

Definition. A *Coxeter group* is a group W generated by a set S with a presentation of the form

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s, t \in S \text{ with } m(s,t) \neq \infty \rangle$$

where $m : S \times S \to \{1, 2, 3, \dots\} \cup \{\infty\}$ is a map with $m(s,t) = m(t,s)$ and $m(s,s) = 1$ for all $s, t \in S$.

The pair (W, S) is a *Coxeter system*.

Given (W, S), the function m can be recovered by setting $m(s,t)$ equal to the order of st.

Theorem. Each finite reflection group W is a finite Coxeter group with respect to the generating set $S = \{s_\alpha : \alpha \in \Delta\}$ of simple reflections corresponding to any simple system Δ in any root system whose reflections generate W.

A *weighted graph* consists of a set of vertices V, a set of edges E given by subsets of V of size two, and a map assigning a weight to each edge.

The *Coxeter graph/diagram* of a Coxeter system (W, S) is the weighted graph with vertex set S, and with an edge labeled by $m(s,t)$ from s to t whenever $s, t \in S$ are such that $m(s,t) > 2$. Any weighted graph whose edge weights belong to $\{3, 4, 5, \dots\} \cup \{\infty\}$ arises as the Coxeter graph of some Coxeter system.

Since edge weights of 3 occur quite often, they are usually omitted. So any unlabeled edge in a graph we draw tacitly is assigned the weight 3.

Example. The Coxeter graph of S_n, which is a Coxeter group with respect to the generating set $\{s_1, s_2, \dots, s_{n-1}\}$ where $s_i = (i, i+1)$, is

$$s_1 \text{ --- } s_2 \text{ --- } \dots \text{ --- } s_{n-1}.$$

Example. The Coxeter graph of $B_n \subset S_{2n}$, which is a Coxeter group with respect to the generating set $\{t_1, t_2, \dots, t_{n-1}\}$ where $t_i = s_i s_{2n-i}$ for $i < n$ and $t_n = s_n$, is

$$t_1 \text{ --- } t_2 \text{ --- } \dots \text{ --- } t_{n-1} \overset{4}{\text{ --- }} t_n.$$

Remark. Two Coxeter groups which are isomorphic as abstract groups may nevertheless have non-isomorphic Coxeter graphs. We saw an example of this on the second homework assignment.

Given a Coxeter system (W, S) and a subset J, define $W_J = \langle s \in J \rangle \subset W$.

Proposition. Suppose W is a finite reflection group generated by $S = \{s_\alpha : \alpha \in \Delta\}$ where Δ is a simple system in some root system $\Phi \subset V$. If $J \subset S$, then

(a) $\Delta_J = \{\alpha \in \Delta : s_\alpha \in J\}$ is a simple system in $\Phi_J = \Phi \cap \mathbb{R}\Delta_J$, which is a root system.

(b) W_J is a reflection group whose length function with respect to Δ_J is the restriction of the length function of W with respect to Δ.

In particular, (W_J, J) is a Coxeter system.

A similar statement holds without the hypothesis that W be a finite reflection group.

We call W_J a *(standard) parabolic subgroup*.

A Coxeter system (W, S) is *irreducible* if its graph is connected.

Proposition. If (W, S) is a Coxeter system which is not irreducible, then W is the direct product of the parabolic subgroups which correspond to its graph's connected components.

Thus we can understand all finite reflection groups by identifying the finite reflection groups which are irreducible. The definitions in the last few lectures give us a convenient language for describing precisely what these are:

Theorem. A Coxeter group is an irreducible finite reflection group if and only if its Coxeter graph belongs to one of the following four infinite families or 6 exceptions:
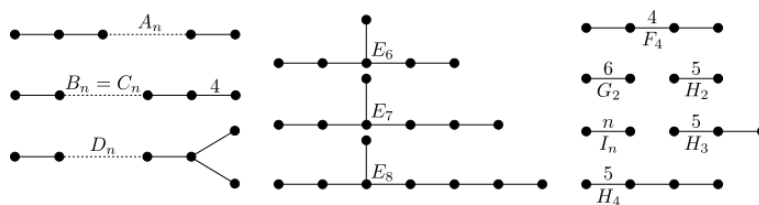
Note that the graph of type $I_2(p)$ is the same as A_2, B_2, H_2, G_2 for $p = 3, 4, 5, 6$.

To ultimately prove this (which we won't do today), we will study the *a priori* harder problem of classifying the finite irreducible Coxeter groups. Later we will see that this classification consists entirely of reflection groups.

2　General theory of Coxeter systems

We now begin to develop the theory of general Coxeter systems in some detail. Most of this theory will be inspired by, and greatly generalize, our results for finite reflection groups.

Let (W, S) be a Coxeter system. There is no need to require that S be a finite set, though most of the examples we encounter will have this property.

Proposition. The unique map $\mathrm{sgn} : S \to \{-1\}$ extends to a homomorphism $W \to \{\pm 1\}$.

It follows that $|W| \geq 2$.

Proof. Observe that $((-1)(-1))^{m(s,t)} = (+1)^{m(s,t)} = 1$.　　　　　□

Define the *length function* of (W, S) exactly as for reflection groups: let this be the map $\ell : W \to \mathbb{N}$ which assigns to w the smallest integer $r \geq 0$ such that $w = s_1 s_2 \cdots s_r$ for some $s_i \in S$.

Call $w = s_1 s_2 \cdots s_r$ a *reduced expression for w* if $s_i \in S$ and $\ell(w) = r$.

We have all of the usual properties of length functions (for $u, v, w \in W$):

- $\ell(1) = 0$.
- $\ell(w) = \ell(w^{-1})$.
- $\ell(uv) \leq \ell(u) + \ell(v)$.
- $\ell(uv) \geq \ell(u) - \ell(v)$.
- $|\ell(ws) - \ell(w)| = 1$ for all $s \in S$.

The last property implies that $\mathrm{sgn}(w) = (-1)^{\ell(w)}$ for $w \in W$.

We need a better, more geometric interpretation of ℓ. To get this, we will identify W as a group generated by "reflections" in a vector space V with a symmetric bilinear form. In contrast to our setup for reflection

groups, the form will no longer be required to be positive definite, and so the analogy between the maps we call reflections and the usual idea of a reflection in Euclidean space will be somewhat weaker.

Given (W, S), define V as the real vector space $\mathbb{R}$-span$\{\alpha_s : s \in S\}$. Here, each α_s is just a formal symbol.

We endow V with the bilinear form defined by

$$(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t)) \text{ for } s, t \in S.$$

Each α_s is a unit vector with respect to this form.

Two vectors α_s and α_t are orthogonal iff $m(s,t) = 2$.

Moreover, we have $(\alpha_s, \alpha_t) \leq 0$ if $s \neq t$.

For $s \in S$ define $\sigma_s : V \to V$ as the linear map

$$\sigma_s v = v - 2(\alpha_s, v)\alpha_s.$$

Proposition. Let $s \in S$ and $u, v \in V$.

   (a) $\sigma_s \alpha_s = -\alpha_s$.

   (b) If $(\alpha_s, v) = 0$ then $\sigma_s v = v$.

   (c) $(\sigma_s u, \sigma_s v) = (u, v)$.

Proof. The proof is the same as for reflection groups. $\square$

Lemma. If $L = \mathbb{R}\alpha_s$ and $U = \{u \in V : (\alpha_s, u) = 0\}$ then $V = L \oplus U$.

Proof. We need to show that $L \cap U = 0$, but this is obvious as $(\alpha_s, \lambda \alpha_s) = \lambda$ for all $\lambda \in \mathbb{R}$. To see that $V = L + U$, note that if $v \in V$ then $v = u + \lambda \alpha_s$ for $u = v - (\alpha_s, v)\alpha_s \in U$ and $\lambda = (\alpha_s, v)$. $\square$

Corollary. If $s \in S$ then $\sigma_s^2 = 1$.

As usual, let $m(s,t)$ denote the order of st in W for $s, t \in S$. The rest of today will be spent proving the following theorem:

Theorem. For each $s, t \in S$, it holds that $(\sigma_s \sigma_t)^{m(s,t)} = 1$. Hence the map $S \mapsto \mathrm{GL}(V)$ given by $s \mapsto \sigma_s$ uniquely extends to a homomorphism $\sigma : W \to \mathrm{GL}(V)$.

We call this homomorphism the *geometric representation* of (W, S).

Sometimes, we also refer to the vector space V itself as the geometric representation of (W, S).

Proof. Fix $s, t \in S$ and let $m = m(s,t)$ and $V_{st} = \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$.

Claim. The restriction of $(\cdot, \cdot)$ to V_{st} is positive semidefinite, meaning that (v, v) is always nonnegative. The restricted form is nondegenerate if and only if $m < \infty$.

Proof. If $v = a\alpha_s + b\alpha_t$ then

$$(v, v) = a^2 + b^2 - 2ab\cos(\pi/m) = (a - b\cos(\pi/m))^2 + b^2 \sin^2(\pi/m) \geq 0.$$

If $m = \infty$, then $(v, v) = 0$ whenever $a = b$. If $m < \infty$ and $v \neq 0$, then evidently $(v, v) > 0$. $\square$

Claim. $\sigma_s V_{st} = \sigma_t V_{st} = V_{st}$.

Proof. The definition of σ_s shows that $\sigma_s V_{st} \subset V_{st}$, and the inclusion must be an equality as σ_s is invertible. $\qquad\square$

It therefore makes sense to compute the order of $\sigma_s \sigma_t$ as a linear map $V_{st} \to V_{st}$. There are two cases:

(a) If $m < \infty$ then the restricted form is positive definite so both σ_s and σ_t act as orthogonal reflections. Since $(\alpha_s, \alpha_t) = -\cos(\pi/m) = \cos(\pi - \pi/m)$, the angle between the lines $\mathbb{R}\alpha_s$ and $\mathbb{R}\alpha_t$ is π/m, and it follows from our earlier analysis of the dihedral groups that $\sigma_s \sigma_t$ acts on V_{st} as a rotation by angle $2\pi/m$. Therefore the order of $\sigma_s \sigma_t$ restricted to V_{st} is exactly m.

(b) If $m = \infty$ then $(\alpha_s, \alpha_t) = -1$, and one can show by induction that $(\sigma_s \sigma_t)^k \alpha_s = 2k(\alpha_s + \alpha_t) + \alpha_s$. It follows that $(\sigma_s \sigma_t)^k \neq 1$ for all $k > 0$, so $\sigma_s \sigma_t$ has infinite order.

In case (a), $\sigma_s \sigma_t$ must have order m in $\mathrm{GL}(V)$ since $V = V_{st} \oplus U$ for

$$U = \{u \in V : (u, v) = 0 \text{ for all } v \in V_{st}\},$$

and $\sigma_s \sigma_t$ fixes U pointwise. (Indeed, note that $U \cap V_{st} = 0$ since the form $(\cdot, \cdot)$ is positive definite on V_{st}. We have $V = V_{st} + U$ since $V = \mathbb{R}\alpha_s + (\mathbb{R}\alpha_s)^\perp = \mathbb{R}\alpha_t + (\mathbb{R}\alpha_t)^\perp$.)

We conclude in either cases that $(\sigma_s \sigma_t)^{m(s,t)} = 1$. $\qquad\square$

Next time, we will augment this result by showing that $\sigma : W \to \mathrm{GL}(V)$ is *faithful*, i.e., injective.