1 Last time: Coxeter groups in general

Recall another equivalent definition of a Coxeter system:

Definition. A *Coxeter system* (W, S) is a pair in which

1. W is a group.
2. $S \subset W$ generates W.
3. Every $s \in S$ has $s^2 = 1 \neq s$.
4. The natural map $\langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s, t \in S \text{ with } m(s,t) < \infty \rangle \to W$ is an isomorphism, where $m(s,t)$ denotes the order of $st \in W$ for $s, t \in S$.

We say that W is a *Coxeter group* relative to the set of *simple generators* S.

The *Coxeter graph* of a Coxeter system (W, S) is the weighted graph with vertex set S, and with an edge labeled by $m(s,t)$ from s to t whenever $s, t \in S$ are such that $m(s,t) > 2$.

The *length function* of (W, S) is the map $\ell : W \to \mathbb{N}$ which assigns to w the smallest integer $r \geq 0$ such that $w = s_1 s_2 \cdots s_r$ for some $s_i \in S$.

Call $w = s_1 s_2 \cdots s_r$ a *reduced expression for* w if $s_i \in S$ and $\ell(w) = r$.

Given (W, S), define V as the real vector space $\mathbb{R}\text{-span}\{\alpha_s : s \in S\}$. Here, each α_s is just a formal symbol. Define $(\cdot, \cdot)$ as the bilinear form in V with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t))$ for $s, t \in S$. Each α_s is a unit vector with respect to this form.

Theorem. For each $s, t \in S$, it holds that $(\sigma_s \sigma_t)^{m(s,t)} = 1$. Hence the map $S \mapsto \mathrm{GL}(V)$ given by $s \mapsto \sigma_s$ uniquely extends to a homomorphism $\sigma : W \to \mathrm{GL}(V)$.

We call this homomorphism the *geometric representation* of (W, S).

Note, for $s \in S$:

1. $\sigma_s \alpha_s = -\alpha_s$.
2. σ_s preserves $(\cdot, \cdot)$.
3. $\sigma_s v = 0$ if $(\alpha_s, v) = 0$.
4. $\sigma_s^2 = 1$.

Notation: from now on, we write wv in place of $\sigma_w v$ for the action of $w \in W$ on $v \in V$ under the geometric representation.

2 Root system of a Coxeter group

Let (W, S) be a Coxeter system.

Define the *root system* Φ of (W, S) to be the set $\{w\alpha_s : w \in W,\ s \in S\}$.

Note that

1. Every $\alpha \in \Phi$ has $(\alpha, \alpha) = 1$.
2. $w\Phi = \Phi$ for all $w \in W$.
3. $\Phi = -\Phi$ since if $\alpha = w\alpha_s$ then $ws\alpha_s = -\alpha$.

By definition, every $\alpha \in V$ can be written uniquely as

$$\alpha = \sum_{s \in S} c_s \alpha_s \qquad \text{with } c_s \in \mathbb{R}.$$

Call α *positive* and write $\alpha > 0$ if every $c_s \geq 0$ in this decomposition and $\alpha \neq 0$. Call α *negative* and write $\alpha < 0$ if every $c_s \geq 0$ in this decomposition and $\alpha \neq 0$.

Let $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$ and $\Phi^- = \{\alpha \in \Phi : \alpha < -0\}$.

Also, given $J \subset W$, let $W_J = \langle J \rangle \subset W$ and define $\ell_J : W_J \to \mathbb{N}$ as the map which assigns to $w \in W_J$ the least integer $r \geq 0$ such that $w = s_1 \cdots s_r$ for some $s_i \in J$.

Note that $\ell(w) \leq \ell_J(w)$ for all $w \in W_J$. Later, we will see that this inequality is an equality.

We arrive to today's main new result:

Theorem. Let $w \in W$ and $s \in S$.

(1) If $\ell(ws) > \ell(w)$ then $w\alpha_s > 0$.

(2) If $\ell(ws) < \ell(w)$ then $w\alpha_s < 0$.

Proof. Note that (1) $\Rightarrow$ (2) since if $v = ws$ then $\ell(ws) < \ell(w) \Leftrightarrow \ell(vs) > \ell(v)$ and $w\alpha_s < 0 \Leftrightarrow v\alpha_s > 0$. We prove (1) by induction on $\ell(w)$. Assume $\ell(ws) > \ell(w)$. If $\ell(w) = 0$ then $w = 1$ and $w\alpha_s = \alpha_s > 0$. Suppose $\ell(w) > 0$ and that w has a reduced expression ending in $t \in S$. Then $\ell(wt) < \ell(w)$ so $s \neq t$.

Let $J = \{s, t\}$, and consider

$$A = \{v \in W : v^{-1}w \in W_J \text{ and } \ell(v) + \ell_J(v^{-1}w) = \ell(w)\}.$$

Note that $w \in A$ so that A is not empty. We may therefore choose $v \in A$ with minimal length.

Write $v_J = v^{-1}w \in W_J$. Then $\ell(w) = \ell(v) + \ell_J(v_J)$ by definition. Note that $wt \in A$ since $(tw^{-1})w = t \in W_J$ and $\ell(wt) + \ell_J(t) = (\ell(w) - 1) + 1 = \ell(w)$. Therefore we must have $\ell(v) \leq \ell(wt) = \ell(w) - 1$.

If $\ell(vs) < \ell(v)$ then we would have

$$\begin{aligned}
\ell(w) &\leq \ell(vs) + \ell((sv^{-1})w) \\
&\leq \ell(vs) + \ell_J(sv^{-1}w) \\
&= \ell(v) - 1 + \ell_J(sv^{-1}w) \\
&\leq \ell(v) - 1 + \ell_J(v^{-1}w) + 1 = \ell(v) + \ell_J(v^{-1}w) = \ell(w)
\end{aligned}$$

in which case all inequalities would have to be equalities and we would have $\ell(w) = \ell(vs) + \ell_J((sv^{-1})w)$ so $vs \in A$. But this would contradict the minimality of $\ell(v)$.

Therefore $\ell(vs) > \ell(v)$, so by induction $v\alpha_s > 0$. A similar argument shows that $\ell(vt) > \ell(v)$ so by induction $v\alpha_t > 0$. As $w = vv_J$, the theorem will be an immediate consequence of the following lemma:

Lemma. $v_J\alpha_s = c_s\alpha_s + c_t\alpha_t$ where $c_s \geq 0$ and $c_t \geq 0$.

Proof. We claim that $\ell_J(v_Js) \geq \ell_J(v_J)$. This follows since if $\ell_J(v_Js) < \ell_J(v_J)$ then

$$\ell(ws) = \ell(vv^{-1}ws) \leq \ell(v) + \ell(v^{-1}ws) = \ell(v) + \ell(v_Js) \leq \ell(v) + \ell_J(v_Js) < \ell(v) + \ell_J(v_J) = \ell(w)$$

but $\ell(w) < \ell(ws)$. Therefore any reduced expression for v_J in W_J must be an alternating product of the factors s, t ending in t. There are two cases to consider:

(a) If $m(s,t) = \infty$ then it is a straightforward exercise in algebra to show that $v_J\alpha_s = a\alpha_s + b\alpha_t$ where $a, b \geq 0$ are integers with $|a - b| = 1$.

(b) Suppose $m = m(s,t) < \infty$. We must have $\ell_J(v_J) < m$ since the unique element of W_J with length m has reduced expressions ending in both s and t. Therefore $v_J = (st)^k$ or $v_J = t(st)^k$ for some $k < m/2$. Observe that in the plane spanned by α_s, α_t in $\mathbb{R}^n$, the vectors α_s and α_t make an angle of $\pi - \pi/m$ and st acts as a rotation by angle $2\pi/m$. By drawing the right picture (try to do this!) one deduces that $v_J\alpha_s$ is in the positive cone spanned by α_s and α_t, so the lemma again holds.

□

□

The theorem has two important corollaries.

Corollary. The root system Φ is the disjoint union of Φ^+ and Φ^-.

This result shows that when Φ is a finite set, it is a root system according to our earlier definition for finite reflection groups.

Proof. Certainly $\Phi^+ \cap \Phi^- = \varnothing$, and if $\alpha = w\alpha_s \in \Phi$ for $w \in W$ and $s \in S$ then either $\alpha \in \Phi^+$ if $\ell(ws) > \ell(w)$ or $\alpha \in \Phi^-$ if $\ell(ws) < \ell(w)$. □

Corollary. The geometric representation $\sigma : W \to \mathrm{GL}(V)$ is faithful, that is, injective.

Proof. Let w belong to the kernel of σ, so that $w\alpha = \alpha$ for all $\alpha \in V$. If $w \neq 1$ then for some $s \in S$ we have $\ell(ws) < \ell(w)$. But the theorem then implies that $w\alpha_s < 0$, contradicting our assumption that $w\alpha_s = \alpha_s > 0$. Therefore σ has trivial kernel, so is an injective homomorphism. □

As an application of this last result, we can clear up a technical property of parabolic subgroups.

As usual, let (W, S) be a Coxeter system. Suppose $J \subset S$.

The *parabolic subgroup* corresponding to J is $W_J = \langle s \in J \rangle \subset W$.

By restricting $m : S \times S \to \{1, 2, 3, \dots\} \cup \{\infty\}$ to $J \times J$, we may define a Coxeter group

$$\overline{W_J} = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s, t \in J \text{ with } m(s,t) < \infty \rangle.$$

Clearly $(\overline{W_J}, J)$ is a Coxeter system, and there is a surjective homomorphism

$$\overline{W_J} \to W_J.$$

Proposition. This map is actually an isomorphism, so we can regard (W_J, J) as a Coxeter system.

Proof. Let $V_J = \mathbb{R}\text{-span}\{\alpha_s : s \in J\} \subset V$ and let $\overline{V_J}$ be the geometric representation of $\overline{W_J}$.

Consider the diagram

$$\begin{array}{ccc} \overline{W_J} & \longrightarrow & \mathrm{GL}(\overline{V_J}) \\ \downarrow & & \uparrow \phi \\ W_J & \longrightarrow & \mathrm{GL}(V_J) \end{array}$$

where the horizontal arrows are the geometric representation of $\overline{W_J}$ and W (restricted to W_J), where $\overline{W_J} \to W_J$ is the surjective map given above, and where ϕ is the isomorphism $\mathrm{GL}(\overline{V_J}) \to \mathrm{GL}(V_J)$ induced by the obvious identification of $V_J \cong \overline{V_J}$.

This diagram is commutative (consider the images of $s \in J$), so as the map $\overline{W_J} \to \mathrm{GL}(\overline{V_J})$ is injective by the previous corollary, the map $\overline{W_J} \to W_J$ must also be injective. □

Next time: more properties of parabolic subgroups, a geometric interpretation of the length function of W, and the strong exchange condition.