1 Last time: the geometric representation is faithful

Let (W, S) be a Coxeter system.

Write $m(s,t)$ for the order of st in W for $s, t \in S$.

Note that $m(s,s) = 1$ and $m(s,t) = m(t,s)$ for all s,t, and we have

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s,t \in S \text{ with } m(s,t) < \infty \rangle.$$

Define V as the real vector space with a basis given by the set of formal symbols $\{\alpha_s : s \in S\}$.

Define $(\cdot,\cdot)$ as the bilinear form on V with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t))$ for $s,t \in S$. Note that $(\alpha_s, \alpha_s) = 1$.

Let $s \in S$ act on V by the formula

$$sv = v - 2(\alpha_s, v)\alpha_s \qquad \text{for } v \in V.$$

Theorem. The map $S \mapsto \mathrm{GL}(V)$ defined by this formula has a unique extension to a homomorphism $W \to \mathrm{GL}(V)$. Thus, setting $wv = s_1(s_2(s_3(\cdots(s_k v)\cdots)))$ for $v \in V$ and $w \in W$, where $w = s_1 s_2 \cdots s_k$ is any expression for w with $s_i \in S$, makes V into a W-module.

We call the W-module V the *geometric representation* if (W, S). One must be careful with this terminology, since the same term is sometimes used to refer to other natural representations of W. The most important properties of this representation, established over the last few lectures, are:

Proposition. It holds that $(wu, wv) = (u, v)$ for all $u, v \in V$ and $w \in W$.

Theorem. If $wv = v$ for all $v \in W$ then $w = 1$.

The geometric representation therefore defines an injective homomorphism $W \to \mathrm{GL}(V)$.

Let $\ell : W \to \mathbb{N}$ denote the length function of (W, S), so that $\ell(w)$ is the least integer $r \geq 0$ such that $w = s_1 \cdots s_r$ for some $s_i \in S$. For any $\alpha \in V$, there is a unique expansion

$$\alpha = \sum_{s \in S} c_s \alpha_s$$

for some real coefficients $c_s \in \mathbb{R}$. Write $\alpha > 0$ if $\alpha \neq 0$ and every $c_s \geq 0$. Write $\alpha < 0$ if $-\alpha > 0$. Another useful fact proved last time:

Theorem. Let $w \in W$ and $s \in S$.

(a) If $\ell(ws) > \ell(w)$ then $w\alpha_s > 0$.

(b) If $\ell(ws) < \ell(w)$ then $w\alpha_s < 0$.

2 Parabolic subgroups

Suppose $J \subset S$. We then have a subgroup $W_J = \langle s \in J \rangle \subset W$.

At the same time, we can define a Coxeter group

$$\overline{W_J} = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s,t \in J \text{ with } m(s,t) < \infty \rangle.$$

Clearly $(\overline{W_J}, J)$ is a Coxeter system, and there is a surjective homomorphism

$$\overline{W_J} \to W_J.$$

Last lecture, we proved that this map is actually an isomorphism, so (W_J, J) as a Coxeter system.

Let $\ell_J : W_J \to \mathbb{N}$ be the length function of (W_J, J), so that $\ell_J(w)$ is the smallest integer $r \geq 0$ such that $w = s_1 \cdots s_r$ for some $s_i \in J$. This is only defined for $w \in W_J$, and clearly $\ell(w) \leq \ell_J(w)$.

Recall that if $w \in W$ then $w = s_1 \cdots s_r$ is a *reduced expression* if $s_i \in S$ and $\ell(w) = r$.

Theorem. The following properties hold:

(a) If $w = s_1 \cdots s_r$ $(s_i \in S)$ is a reduced expression for $w \in W_J$ then every factor $s_i \in J$.

Therefore $\ell_J(w) = \ell(w)$ for $w \in W_J$.

(b) Let $I, J \subset S$. Then $I \subset J$ if and only if $W_I \subset W_J$, and $W_I \cap W_J = W_{I\cap J}$.

(c) The set S is a minimal generating set for W.

Proof. To prove (a), we use induction of $\ell(w)$, noting that $\ell(1) = 0 = \ell_J(1)$. Assume $w \neq 1$ and $w = s_1 \cdots s_r$ is a reduced expression, and set $s = s_r$. Then $w\alpha_s > 0$ by the theorem proved last time. Since $w \in W_J$, we can write $w = t_1 \cdots t_q$ for some $t_i \in J$. One checks that

$$w\alpha_s = t_1 \cdots t_q \alpha_s = \alpha_s + \sum_{i=1}^{q} c_i \alpha_{t_i} \qquad \text{for some coefficients } c_i \in \mathbb{R}.$$

Since $w\alpha_s < 0$, we must have $s = t_i \in J$ for some i, and in this case $ws_r = s_1 \cdots s_{r-1} \in W_J$ is also a reduced expression, so by induction $s_i \in J$ for $1 \leq i < r$.

For (b), note that $W_I \cap S = I$ by part (a) since any expression of length one is reduced. Therefore if $W_I \subset W_J$ then $I = W_I \cap S \subset W_J \cap S = J$. It clearly holds that $W_{I\cap J} \subset W_I \cap W_J$ and the reverse containment follows by part (a).

Finally, note that if $I \subset S$ and $W = W_I$ then part (b) implies that $S \subset I$ so $I = S$. $\square$

3 Geometric interpretation of the length function

Recall that the root system of (W, S) is the set of vectors $\Phi = \{w\alpha_s : w \in W,\ s \in S\} \subset V$.

Define $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$ and $\Phi^- = \{\alpha \in \Phi : \alpha < 0\}$.

The last theorem in the first section today implies that:

Corollary. Φ is the disjoint union of Φ^+ and Φ^-.

The following generalizes a fact we saw earlier for finite reflection groups:

Proposition. If $s \in S$ then $s\alpha_s = -\alpha_s$, and $\alpha \mapsto s\alpha$ defines a permutation of $\Phi^+ - \{\alpha_s\}$.

Proof. Suppose $\alpha \in \Phi^+$ and $\alpha \neq \alpha_s$. Since all elements of Φ are unit vectors, we have $\alpha \notin \mathbb{R}\alpha_s$, so $\alpha = \sum_{t\in S} c_t\alpha_t$ where each coefficient $c_t \geq 0$, and $c_q > 0$ for some $q \neq s$. We cannot have $s\alpha \in \Phi^-$ since if $s\alpha = \sum_{t\in S} c'_t\alpha_t$ then $c_t = c'_t$ for all $t \neq s$, and in particular $c'_q = c_q > 0$. At the same time, clearly $s\alpha \notin \mathbb{R}\alpha_s$, so $s\alpha$ must belong to $\Phi^+ - \{\alpha_s\}$. Since s acts as an invertible map, the results follows. $\square$

We may now characterize the length of $w \in W$ in terms of positive and negative roots, much like for finite reflection groups.

Proposition. If $w \in W$ then $\ell(w)$ is the number of positive roots $\alpha \in \Phi^+$ with $w\alpha \in \Phi^-$.

Proof. Let $\Pi(w)$ be the set of positive roots $\alpha \in \Phi^+$ with $w\alpha \in \Phi^-$ and set $n(w) = |\Pi(w)|$. The proof is the same as in the reflection group case several lectures ago. One first verifies for $s \in S$ and $w \in W$ that

$$w\alpha_s > 0 \Rightarrow n(ws) = n(w) + 1$$

$$w\alpha_s < 0 \Rightarrow n(ws) = n(w) - 1$$

using the previous proposition. Comparing these properties to the identical ones pertaining to $\ell(w)$, one deduces that $n(w) = \ell(w)$ by induction. □

4 Roots and reflections

By the definition of the geometric representation, each $s \in S$ acts on V as a reflection.

More generally, we can associate a reflection to any $\alpha \in \Phi$ as follows.

Proposition. If $\alpha \in \Phi$ then the set $\{wsw^{-1} : \alpha = w\alpha_s \text{ for } w \in W \text{ and } s \in S\}$ contains exactly one element. I.e., if $\alpha = w\alpha_s$ then $wsw^{-1} \in W$ depends only on α, not on the choice of $w \in W$ and $s \in S$.

Proof. For $v \in V$, we compute that

$$\begin{aligned} wsw^{-1}v &= w\left(w^{-1}v - 2(w^{-1}v, \alpha_s)\alpha_s\right) \\ &= v - 2(w^{-1}v, \alpha_s)w\alpha_s \\ &= v - 2(v, w\alpha_s)w\alpha_s = v - 2(v, \alpha)\alpha. \end{aligned}$$

The result now follows since W acts faithfully on V. □

Given $\alpha \in \Phi$, define $s_\alpha = wsw^{-1}$ where $w \in W$ and $s \in S$ are any elements with $\alpha = w\alpha_s$. The proposition shows that this construction is well-defined. Note that $s_{\alpha_s} = s$ for $s \in S$.

Let $T = T(W, S) = \{s_\alpha : \alpha \in \Phi\}$.

Example. If $W = S_n$ and $S = \{s_i = (i, i+1) \in S_n : i \in [n-1]\}$ then

$$T = \{w(i, i+1)w^{-1} : i \in [n-1],\ w \in W_n\} = \{t_{ij} = (i, j) \in S_n : 1 \le i < j \le n\}.$$

The set T is naturally indexed by Φ^+.

Proposition. The map $\alpha \mapsto s_\alpha$ is a bijection $\Phi^+ \to T$.

Proof. If $s_\alpha = s_\beta$ for $\alpha, \beta \in \Phi^+$ then $v - 2(v, \alpha)\alpha = v - 2(v, \beta)\beta)$ for all $v \in V$, so taking $v = \beta$ gives $\beta = (\beta, \alpha)\alpha$. Applying $(\beta, \cdot)$ to both sides of this equation gives $(\beta, \alpha)^2 = 1$, so $\beta \in \{\pm\alpha\}$. But as both roots are positive, necessarily $\alpha = \beta$. □

We note another easy lemma for use later.

Lemma. If $\alpha, \beta \in \Phi$ and $\beta = w\alpha$ for some $w \in W$ then $ws_\alpha w^{-1} = s_\beta$.

Proof. Suppose $g \in W$ is such that $g\alpha_s = \alpha$. Then $ws_\alpha w^{-1} = wgs(wg)^{-1}$. Since $wg\alpha_s = w\alpha = \beta$ it follows that $(wg)s(wg)^{-1} = s_\beta$. □

This leads us to the following generalization of our earlier theorem:

Proposition. Let $w \in W$ and $\alpha \in \Phi^+$. Then $\ell(ws_\alpha) > \ell(w)$ if and only if $w\alpha > 0$.

Proof. It suffices to show that if $\ell(ws\alpha) > \ell(w)$ then $w\alpha > 0$. (Why is this enough?)

We proceed by induction of $\ell(w)$. The case when $\ell(w) = 0$ is clear, since then $w = 1$.

Assume $\ell(w) > 0$, so that $\ell(sw) < \ell(w)$ for some $s \in S$. We then have $\ell((sw)s_\alpha) = \ell(s(ws_\alpha)) \geq \ell(ws_\alpha) - 1 > \ell(w) - 1 = \ell(sw)$, so by induction $sw\alpha > 0$.

Now suppose $w\alpha < 0$. The only negative root made positive by s is $-\alpha_s$, so $w\alpha = -\alpha_s$. But then $sw\alpha = s(-\alpha_s) = \alpha_s$ so $(sw)s_\alpha(sw)^{-1} = s$ and $ws_\alpha = sw$. This is impossible since $\ell(ws_\alpha) > \ell(w) > \ell(sw)$.

We deduce by this contradiction that instead $w\alpha > 0$. ☐

5 Strong exchange condition

We may now prove the most important technical property of a Coxeter group, generalizing the exchange condition that we encountered for finite reflection groups.

Theorem. Let $w = s_1 \cdots s_r$ ($s_i \in S$) with $\ell(w) \leq r$. Suppose $t \in T$ is such that $\ell(wt) < \ell(w)$.

Then there exists an index $i \in [r]$ such that $wt = s_1 \cdots \widehat{s_i} \cdots s_r$.

If $\ell(w) = r$, then the index i is unique.

Proof. Let $t = s_\alpha$ for $\alpha \in \Phi^+$. Since $\ell(wt) < \ell(w)$, we have $w\alpha < 0$. As $\alpha > 0$, there must exist an index $i \leq r$ such that $s_{i+1} \cdots s_r\alpha > 0$ but $s_i s_{i+1} \cdots s_r \alpha < 0$. Since α_{s_i} is the only positive root which s_i makes negative, it must hold that $s_{i+1} \cdots s_r \alpha = \alpha_{s_i}$. But our lemma above, it follows that $(s_{i+1} \cdots s_r)t(s_{i+1} \cdots s_r)^{-1} = s_i$. Thus

$$wt = (s_1 \cdots s_i)(s_{i+1} \cdots s_r)t = (s_1 \cdots s_i)s_i(s_{i+1} \cdots s_r) = s_1 \cdots \widehat{s_i} \cdots s_r.$$

If $\ell(w) = r$, then the index i must be unique since if we had

$$wt = s_1 \cdots \widehat{s_i} \cdots s_r = s_1 \cdots \widehat{s_j} \cdots s_r$$

for some $1 \leq i < j \leq n$ then it would follow that $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_r$, which is impossible. ☐

When $t \in S \subset T$, the theorem is referred to more simply as the *exchange condition*.

Next time: applications of the exchange condition, and an introduction to the Bruhat order.