1 Summary of facts so far

By now we have developed quite a number of general, useful properties of Coxeter systems.

Let (W, S) be a Coxeter system. Write $m(s, t)$ for the order of st in W for $s, t \in S$, so that

$$W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s, t \in S \text{ with } m(s, t) < \infty \rangle.$$

Define $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$ as the real vector space with a basis indexed by S, and define $(\cdot, \cdot)$ as the bilinear form on V with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s, t))$ for $s, t \in S$.

For any $\alpha \in V$, there is a unique expansion

$$\alpha = \sum_{s \in S} c_s \alpha_s$$

for some real coefficients $c_s \in \mathbb{R}$. Write $\alpha > 0$ if $\alpha \neq 0$ and every $c_s \geq 0$. Write $\alpha < 0$ if $-\alpha > 0$.

The *root system* of (W, S) is the set $\Phi = \{w\alpha_s : w \in W,\ s \in S\} \subset V$.

Let $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$ and $\Phi^- = \{\alpha \in \Phi : \alpha < 0\}$.

The following five theorems summarize most of our progress in the last few lectures.

Theorem (Geometric representation)**.** The vector space V has a unique W-module structure in which $s \in S$ acts as the reflection $s : v \mapsto v - 2(\alpha_s, v)\alpha_s$ for $v \in V$. With respect to this structure, it holds that $(wu, wv) = (u, v)$ for all $w \in W$ and $u, v \in V$. If $w \in W$ is such that $wv = v$ for all $v \in V$ then $w = 1$.

Theorem (Parabolic subgroups)**.** Let $J \subset S$ and define $W_J = \langle s \in J \rangle \subset W$

(i) (W_J, J) is a Coxeter system.

(ii) The length function of W_J (defined with respect to J) agrees with the length function of W.

(iii) The set S is a minimal generating set of W.

(iv) If $w \in W_J$ and $w = s_1 \cdots s_r$ ($s_i \in S$) is a reduced expression, then every $s_i \in J$.

Theorem (Geometric interpretation of length function)**.** The following properties hold:

(a) If $w \in W$ and $s \in S$ then $\ell(ws) < \ell(w)$ if and only if $w\alpha_s < 0$.

(b) There is a disjoint union $\Phi = \Phi^+ \sqcup \Phi^-$.

(c) If $s \in S$ then $s\alpha_s = -\alpha_s$ and $s\beta \in \Phi^+$ for all $\alpha \neq \beta \in \Phi^+$.

(d) The length $\ell(w)$ of $w \in W$ is the number of root $\alpha \in \Phi^+$ with $w\alpha \in \Phi^-$.

Theorem (Reflections)**.** The following properties hold:

(a) If $w\alpha_s = w'\alpha_{s'} = \alpha \in \Phi$ for some $w, w' \in W$ and $s, s' \in S$, then $wsw^{-1} = w's'(w')^{-1}$.

Denote this element by $s_\alpha \in W$, and let $T = \{s_\alpha : \alpha \in \Phi\} = \{wsw^{-1} : w \in W,\ s \in S\}$.

(b) The correspondence $\alpha \mapsto s_\alpha$ is a bijection $\Phi^+ \to T$.

(c) If $\alpha, \beta \in \Phi$ and $\beta = w\alpha$ for some $w \in W$ then $ws_\alpha w^{-1} = s_\beta$.

(d) If $w \in W$ and $\alpha \in \Phi^+$ then $\ell(ws_\alpha) > \ell(w)$ if and only if $w\alpha > 0$.

Theorem (Strong exchange condition)**.** Let $w = s_1 \cdots s_r$ ($s_i \in S$) with $\ell(w) \leq r$. Suppose $t \in T$ is such that $\ell(wt) < \ell(w)$. Then there exists an index $i \in [r]$ such that $wt = s_1 \cdots \widehat{s_i} \cdots s_r$. If $\ell(w) = r$, then the index i is unique.

2 Bruhat order

This will be the most useful partial order on W which is compatible with $\ell : W \to \mathbb{N}$.

Definition. The *Bruhat order* on W is partial order $<$ which is the transitive closure of the relation with $w < wt$ whenever $w \in W$ and $t \in T$ and $\ell(wt) < \ell(w)$.

This means that if $u, v \in W$ then $u < v$ if and only if there are reflections $t_1, t_2, \dots, t_k \in T$ with $v = ut_1t_2 \cdots t_k$ and $\ell(u) < \ell(ut_1) < \ell(ut_1t_2) < \cdots < (ut_1t_2 \cdots t_k) = \ell(v)$.

Remark. Suppose $<_2$ is the transitive closure of relation on W with $w <_2 tw$ for $w \in W$ and $t \in T$ and $\ell(w) < \ell(tw)$. Then $< = <_2$ since

$$w <_2 tw \quad \Leftrightarrow \quad w < wt' \qquad \text{for } t' = w^{-1}tw \in T.$$

Thus despite appearances the definition of the Bruhat order is symmetric between left and right.

The *left/right weak order* on W is the partial order defined in the same way but requiring $t \in S \subset T$. This order is sometimes useful, but we won't spend much time discussing it.

Note, if $w < wt$ for $w \in W$ and $t \in T$ then $\ell(wt) > \ell(w)$ but it is not required that $\ell(wt) = \ell(w) + 1$. Also, observe that if $s \in S$ then $w < ws$ if and only if $\ell(w) < \ell(ws)$.

Example. If $|S| = 2$ so that W is a dihedral group and $u, v \in W$, then $u \leq v$ if and only if $\ell(u) \leq \ell(v)$. (Try to work this our for yourself!)

Example. Let $W = S_n$ and $S = \{(i, i+1) : i = 1, 2, \dots, n-1\}$ so that $T = \{(i, j) : 1 \leq i < j \leq n\}$. We have $\ell(w(i,j)) > \ell(w)$ if and only if $w(i) > w(j)$ for $i < j$. Therefore $u < v$ in the Bruhat order of S_n if and only if the sequence

$$v(1)v(2)\cdots v(n)$$

can be obtained from

$$u(1)u(2)\cdots u(n)$$

by a sequence of moves in which we switch numbers in positions i and j (with $i < j$) with the ith number greater than the jth number.

For example, **24**153 $\to$ 4**2**15**3** $\to$ **45**123 $\to$ 54123 so 24153 $<$ 54123.

This doesn't suggest a very efficient algorithm for checking whether $u < v$ for $u, v \in S_n$. The following result of Deodhar, which we quote without proof, provides such an algorithm.

Proposition (Deodhar)**.** If $(a_1, \dots, a_k)$ is a sequence of integers, then write $[a_1, \dots, a_k]$ for the sequence rewritten in increasing order. Define $(a_1, \dots, a_k) \preceq (b_1, \dots, b_k)$ if $a_i \leq b_i$ for all i.

Then $u \leq v$ in S_n if and only if $[u(1), \dots, u(k)] \preceq [v(1), \dots, v(k)]$ for $1 \leq k \leq n$.

Let (W, S) be an arbitrary Coxeter system.

Proposition. Let $u \leq v$ and $s \in S$. Then $us \leq v$ or $us \leq vs$ (or both).

This turns out to be a key technical property, which is sometimes called the *lifting property*.

Proof. If suffices to assume that $v = ut$ with $t \in T$ and $\ell(v) > \ell(u)$. If $s = t$ then there is nothing to prove since then $v = us$. Assume $s \neq t$. There are two cases to consider:

(a) If $\ell(us) = \ell(u) - 1$ then $us < u < v$ so $us \leq v$.

(b) Suppose $\ell(us) = \ell(u) + 1$. Then $ust' = vs$ for $t' = sts$, so it suffices to show that $\ell(us) < \ell(vs)$, which will imply that $us \leq vs$.

We argue by contradiction, so suppose $\ell(vs) < \ell(us)$. We now apply the strong exchange condition as follows. Note that for any reduced expression $u = s_1 \cdots s_r$ the expression $us = s_1 \cdots s_r s$ is also reduced as $\ell(us) > \ell(u)$.

Then $vs = ust'$ has a reduce expression given by omitting one factor from

$$s_1 \cdot s_r s.$$

The omitted factor cannot be s since $s \neq t$ and $v = ut$, so we must have $vs = s_1 \cdots \widehat{s_i} \cdots s_r s$ for some i so $v = s_1 \cdots \widehat{s_i} \cdots s_r$ contradicting our assumption that $\ell(v) > \ell(u)$.

This completes the proof of the proposition. □

A *subexpression* of a reduced expression $w = s_1 \cdots s_r$ ($s_i \in S$) is any product of the form $s_{i_1} s_{i_2} \cdots s_{i_q}$ where $1 \leq i_1 < i_2 < \cdots < i_q \leq r$.

Theorem. Let $w = s_1 \cdots s_r$ be a fixed but arbitrary reduced expression for $w \in W$. Then $v \leq w$ in Bruhat order if and only if v can be obtained as a subexpression of the chosen reduced expression for w.

Proof. If $w = vt$ for some $t \in T$, so that $\ell(v) < \ell(w)$, then $\ell(wt) < \ell(w)$ so by the strong exchange condition $v = s_1 \cdots \widehat{s_i} \cdots s_r$ for some i. Since the strong exchange condition does not require reduced expressions, iterating the previous observation implies that if $v < w$ then v is a subexpression of $s_1 \cdots s_r$.

For the other direction, suppose $v = s_{i_1} \cdots v_{i_q}$ where $1 \leq i_1 < i_2 < \cdots < i_q \leq r$. We need to show that $v \leq w$. For this, proceed by induction on $r = \ell(w)$. If $r = 0$ then $v = w = 1$ so $v \leq w$.

Assume $r > 0$ and $i_q < r$. Then, by the inductive hypothesis applied to $s_1 \cdots s_{r-1}$, we get that

$$s_{i_1} \cdots s_{i_q} \leq s_1 \cdots s_{r-1} < s_1 \cdots s_r = w.$$

If $r > 0$ and $i_q = r$, we first deduce by induction that

$$s_{i_1} \cdots s_{i_{q-1}} \leq s_1 \cdots s_{r-1}$$

and then use the lifting property with $s = s_{i_q} = s_r$ to conclude that

$$s_{i_1} \cdots s_{i_{q-1}} s_{i_q} \leq s_1 \cdots s_{r-1} <= w \qquad \text{or} \qquad s_{i_1} \cdots s_{i_{q-1}} s_{i_q} \leq s_1 \cdots s_{r-1} s_r = w.$$

In either case we get $v \leq w$ as desired. □

The characterization of the Bruhat order in the theorem is a much more instructive way of thinking about this order. However, it is awkward as an initial definition because of the apparent dependence on the choice of reduced expression: this makes showing that $<$ is transitive nontrivial.

As an application, we can answer one natural question about the Bruhat order of parabolic subgroups.

Corollary. If $J \subset S$ then the Bruhat order of (W_J, J) agrees with the Bruhat order of (W, S) restricted to W_J.

Proof. If $w \in W_J$ then w has a reduced expression in W involving only factors in J, and by the theorem $v \leq w$ in the Bruhat order of W or W_J if and only if v occurs as a subexpression of this reduced expression. □

Next time: a few more properties of the Bruhat order, and a description of a fundamental domain for W acting on V.