1 More about Bruhat order

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$. Recall the strong exchange principle:

Theorem (Strong exchange condition)**.** Let $w = s_1 \cdots s_r$ ($s_i \in S$) with $\ell(w) \leq r$. Suppose $t \in T$ is such that $\ell(wt) < \ell(w)$. Then there exists an index $i \in [r]$ such that $wt = s_1 \cdots \widehat{s_i} \cdots s_r$. If $\ell(w) = r$, then the index i is unique.

Recall the main definition from last time:

Definition. The *Bruhat order* on W is the partial order $<$ which is the transitive closure of the relation with $w < wt$ whenever $w \in W$ and $t \in T$ and $\ell(wt) < \ell(w)$.

Note that if $s \in S$ and $w \in W$ then $ws < w$ if and only if $\ell(ws) < \ell(w)$. A conceptually more useful and appealing characterization of the Bruhat order is given by the following result proved last time.

Theorem. If $v, w \in W$ then $v \leq w$ if and only if for some (equivalently, every) reduced expression $w = s_1 \cdots s_r$ there are indices $1 \leq i_1 < i_2 < \cdots < i_q \leq r$ such that $v = s_{i_1} s_{i_2} \cdots s_{i_q}$.

Corollary. The Bruhat order of (W_J, J) for $J \subset S$ agrees with the Bruhat order of (W, S).

The following technical property from last time will be of use again today:

Lemma (Lifting Property)**.** If $u, v \in W$ and $u \leq v$ and $s \in S$, then $us \leq v$ or $us \leq vs$ (or both).

For the next proposition, we need a new lemma.

Lemma. Let $v, w \in W$ with $v < w$ and $\ell(w) = \ell(v) + 1$. Suppose $s \in S$ is such that $v < vs$ and $vs \neq w$. Then $w < ws$ and $vs < ws$.

Proof. By the lifting property, we have $vs \leq w$ or $vs \leq ws$. The first case cannot occur since $\ell(vs) = \ell(w)$ but $vs \neq w$. Therefore $vs \leq ws$. As $v \neq w$, we must have $vs < ws$. This implies that $\ell(vs) < \ell(ws)$. As $\ell(vs) = \ell(w)$, it follows that $w < ws$. □

A chain in a partially ordered set is a sequence of elements $a_0, a_1, a_2, \ldots, a_n$ such that $a_0 < a_1 < a_2 < \cdots < a_n$. Such a chain is between two elements a and b if $a = a_0$ and $a_n = b$. We have the following result about chains in W with respect to the Bruhat order.

Proposition. Let $v, w \in W$ with $v < w$. Then there exist $w_0, w_1, \ldots, w_m \in W$ such that $v = w_0 < w_1 < \cdots < w_m = m$ and $\ell(w_i) = \ell(v) + i$ for all i.

Proof. We proceed by induction on the sum $\ell(v) + \ell(w)$ which is also at least one. If $\ell(v) + \ell(w) = 1$ then $v = 1$ and $w \in S$ and the result is trivial.

Assume $\ell(v) + \ell(w) > 1$ and let $w = s_1 \cdots s_r$ be a reduced expression. Set $s = s_r$. Then $v = s_{i_1} \cdots s_{i_q}$ for some indices $1 \leq i_1 < \cdots < i_q \leq r$. There are two cases to consider:

(a) Suppose $v < vs$. We may assume that $i_q < r$ by the exchange condition, since otherwise $s = s_r$ would be a descent of $s_{i_1} \cdots s_{i_{q-1}}$. It follows that v is also a subexpression of $ws < w$, so $v < ws$. By induction one can find a chain of the desired type from v to w and then one more steps gets us to w.

(b) Suppose instead that $vs < v$. By induction we then have a chain in $(W, <)$ of the form

$$vs = w_0 < w_1 < \cdots < w_m = w$$

with $\ell(w_i) = \ell(v) + i$ for all i. Choose i to be the smallest index such that $w_i s < w_i$. Some such index exists since $w_0 s = v > vs = w_0$ but $w_m s = ws < w = w_m$.

Note that if $w_i \neq w_{i-1}s$ then applying the previous lemma to

$$w_{i-1} < w_{i-1}s \neq w_i$$

gives $w_i < w_i s$, contradicting the definition of i. Therefore $w_i = w_{i-1}s$.

For $1 \leq j < i$ we have $w_j \neq w_{j-1}s$ since $w_j < w_j s$. For such j, applying the lemma to

$$w_{j-1} < w_{j-1}s \neq w_j$$

gives $w_{j-1}s < w_j s$. Combining these observations show that

$$v = w_0 s < w_1 s < \cdots < w_{i-1}s = w_i < w_{i+1} < \cdots < w_m = w$$

is a chain in the Bruhat order of W with the desired properties.

□

Note that if $v = w_0 < w_1 < \cdots < w_m = w$ is any chain in the Bruhat order of W then $\ell(w_i) \geq \ell(w_{i-1})+1$ so $\ell(w) \geq \ell(v) + m$ and $m \leq \ell(w) - \ell(v)$. Therefore $\ell(w) - \ell(v)$ is an upper bound on the length of any chain in the Bruhat order of W from v to w. The proposition shows that this upper bound is always achieved. In other words, every maximal chain in the Bruhat order of W between v and w has the same length $\ell(w) - \ell(v)$. This property is equivalent to the following statement.

Corollary. $(W, <)$ is a *graded* partially ordered set with *rank function* ℓ.

2 Minimal length coset representatives

Continue to let (W, S) be a Coxeter system. Let $J \subset S$ and recall that $W_J = \langle s \in J \rangle \subset W$.

Define $W^J = \{w \in W : \ell(ws) > \ell(w) \text{ for all } s \in J\}$.

Proposition. For each $w \in W$ there is a unique $u \in W^J$ and $v \in W_J$ such that $w = uv$. Moreover, it holds for these elements that $\ell(w) = \ell(u) + \ell(v)$. Also, u is the unique element of smallest length in the coset $wW_J = \{wx : x \in W_J\}$.

Proof. The proof via the exchange principle is the same as for the result for reflection groups. □

Corollary. If $u \in W^J$ and $v \in W_J$ then $\ell(uv) = \ell(u) + \ell(v)$.

3 Fundamental domain for W

In this section, we assume that S is a finite set. Let $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$ be the usual W-module on which $s \in S$ acts by $sv = v - 2(\alpha_s, v)\alpha_s$ for $v \in V$, where $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the bilinear form with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t))$.

We want to make the geometry of W's action on V more explicit. This goal is obstructed by the fact that, unlike in the case of finite reflection groups, the bilinear form $(\cdot,\cdot)$ is no longer necessarily non-degenerate. As a substitute, define V^* as the real vector space of $\mathbb{R}$-linear maps $V \to \mathbb{R}$.

Let W act on V^* by defining $w\lambda$ for $w \in W$ and $\lambda \in V^*$ as the linear map with the formula

$$(w\lambda)(v) = \lambda(w^{-1}v) \qquad \text{for } v \in V.$$

Check that this is an action!

For each $s \in S$, define these three sets:

$Z_s = \{f \in V^* : f(\alpha_s) = 0\}$,

$A_s = \{f \in V^* : f(\alpha_s) > 0\}$,

$A'_s = -A_s = \{f \in V^* : f(\alpha_s) < 0\}$.

Let $C = \bigcap_{s \in S} A_s \subset V^*$.

Proposition. If $s \in S$ then $sf = f$ for all $f \in Z_s$.

Proof. Let $s \in S$ and $f \in Z_s$. If $t \in S$ then $(sf)(\alpha_t) = f(s\alpha_t)$ is either $-f(\alpha_s) = 0 = f(\alpha_t)$ if $t = s$, or $f(\alpha_t - 2(\alpha_s, \alpha_t)\alpha_s) = f(\alpha_t)$ by linearity if $t \neq s$. Therefore $sf = f$. □

Let $b_1, \dots, b_n$ be a basis of V with $\alpha_s = b_1$ and $(b_i, \alpha_s) = 0$ for $i > 1$.

Let $f_1, \dots, f_n$ be the dual basis of V^* so that $f_i(b_j) = \delta_{ij}$ for all i, j.

Proposition. It holds that $sf_i = f_i$ for $i > 1$.

Proof. Fix $i > 1$. We have $(sf_i)(b_1) = (sf_i)(\alpha_s) = -f_i(\alpha_s) = 0 = f(b_1)$.

Likewise $(sf_i)(b_j) = f_i(sb_j) = f_i(b_j)$ for $j > 1$. □

We may identify V with $\mathbb{R}^n$ by fixing a basis, e.g., $\{\alpha_s : s \in S\}$, and letting it correspond to the standard basis of $\mathbb{R}^n$. We then identify V^* with $\mathbb{R}^n$ via the associated dual basis.

Proposition. Let $s \in S$. In the standard topology of $\mathbb{R}^n$ under this identification:

(1) Z_s is closed, A_s and A'_s are open, and C is open.

(2) The closure $\overline{A_s}$ of A_s is $A_s \cup Z_s$.

(3) Define D as the closure of C. Then $D = \bigcap_{s \in S} \overline{A_s}$.

Proof. For part (1), note that Z_s is the inverse image of the closed set $\{0\}$ under the linear (continuous) map $f \mapsto f(\alpha_s)$. The sets A_s and A'_s are likewise the inverse images under continuous maps of the open sets $(0, \infty)$ and $(-\infty, 0)$. Part (2) is clear and part (3) follows from part (2). □

Also, note that the action of W on V^* is continuous.

We partition $D = \{f \in V^* : f(\alpha_s) \geq 0 \text{ for all } s \in S\}$ into sets

$$C_J = \left(\bigcap_{s \in J} Z_s \right) \cap \left(\bigcap_{s \notin J} A_s \right)$$

for $J \subset S$. Note that the choice of J just determines which of this $|S|$ inequalities $\leq$ in the definition of D is an equality $=$ or a strict inequality $<$, so the sets C_J are disjoint and form a partition of D. At the extremes, we have $C_\varnothing = C$ and $C_s = \{0\}$.

Since $s \in S$ fixes Z_s pointwise, W_J fixes C_J pointwise. Conversely:

Proposition. If $s \in S$ and $f \in C_J$ and $sf = f$, then $s \in J$.

Proof. Let $s \in S$ and $f \in C_J$. If $s \notin J$ then $f(\alpha_s) > 0$. However, if $sf = f$ then $f(\alpha_s) = (sf)(s\alpha_s) = f(-\alpha_s) = -f(\alpha_s)$ so $f(\alpha_s) = 0$. Therefore $sf = f$ implies $s \in J$. □

Finally, let $U = \bigcup_{w\in W} w(D) = \bigcup_{w\in W}\bigcup_{J\subset} w(C_J)$.

Since D is a *convex cone* (meaning that if $f, g \in D$ and $\theta\in[0,1]$ and $\lambda\in(0,\infty)$ then $\theta f + (1-\theta)g \in D$ and $\lambda f \in D$) the set U must also be at least a cone: this is called the *Tits cone*. It will turn out that this cone is also convex. To prove this, we'll need a lemma.

Lemma. Let $s \in S$ and $w \in W$. Then $\ell(sw) > \ell(w)$ if and only if $w(C) \subset A_s$. Also, $\ell(sw) < \ell(w)$ if and only if $w(C) \subset A'_s$.

Proof. We only prove the first assertion since the second follows similarly. Note that $\ell(sw) > \ell(w)$ is equivalent to $w^{-1}\alpha_s > 0$. In this case, if $f \in C$ then $(wf)(\alpha_s) = f(w^{-1}\alpha_s) > 0$ so $wf \in A_f$. Therefore $\ell(sw) > \ell(w)$ implies $w(C) \subset A_s$. Conversely, if $(wf)(\alpha_s) = f(w^{-1}\alpha_s) > 0$ for all $f \in C$ then, by considering such f which take very small positive values on α_t for $t \neq s$, it follows that $w^{-1}\alpha_s > 0$ so $\ell(sw) > \ell(w)$. □

We now have today's main theorem.

Theorem. The action of W on V^* has the following properties:

(a) Let $w \in W$ and $I, J \subset S$. If $w(C_I) \cap C_J \neq \varnothing$ then $I = J$ and $w \in W_I$ so $w(C_I) = C_I$. Thus W_I is the stabilizer of the each point of C_I in W, and the sets $w(C_I)$ for $w \in W$, $I \subset S$ are disjoint.

(b) The W-orbit of each point in U intersects D in exactly one point.

(c) The cone U is convex and every closed line segment in U intersects finitely many of the sets

$$\mathscr{C} = \{w(C_I) : w \in W,\ I \subset S\}.$$

Proof. We prove part (a) by induction on $\ell(w)$, the case when $w = 1$ being obvious. Assume $\ell(w) > 0$ and write $w = s(sw)$ for some $s \in S$ with $\ell(sw) < \ell(w)$. The lemma forces us to have $w(C) \subset s(A_s) = A'_s$ so by continuity $w(D) \subset \overline{A'_s} = A'_s \cup Z_s$. As $D \subset \overline{A_s}$, we have $w(D) \cap D \subset Z_s$, so s fixes each point in $w(D) \cap D$, and hence also each $f \in C_J \cap w(C_I)$. Two things follow.

First, s fixes some point of C_J so $s \in J$.

Second, $C_J \cap sw(C_I) = s(C_J \cap w(C_I))$ is nonempty.

By induction (replacing w by sw), it follows that $I = J$ and $sw \in W_I$. But now since $s \in J = I$, we have $w = s(sw) \in W_I$ as needed.

To prove (b), note that by the definition of U, each W-orbit in U meets D is at least one point. If $f, g \in D$ both lie in the same W-orbit then $wf = g$ for some $w \in W$. Suppose $f \in C_I$ and $g \in C_J$ so that $w(C_I) \cap C_J \neq \varnothing$. By (a), we then have $I = J$ and $w \in W_I$, so $f = wf = g$.

For (c), let $f, g \in U$. It is enough to prove that the closed line segment $L = \{\theta f + (1-\theta)g : \theta \in [0,1]\}$ is covered by a finite number of sets in $\mathscr{C}$. This is clear is $f, g \in D$ since D is convex and covered by a finite number of the sets C_I. Without loss of generality we may assume that $f \in D$ and $g \in w(D)$.

We proceed by induction on $\ell(w)$. The case $w = 1$ was just covered. Let $\ell(w) > 0$. Then $L \cap D$ is covered by finitely many sets in $\mathscr{C}$. It remains to cover $L \setminus D$. Let $I \subset S$ be such that $g \in A'_s$ for $s \in I$ and $g \in \overline{A_s}$ for $s \notin I$. Let $h \in D$ be the endpoint of $L \setminus D$ distinct from g. If $h \in A_s$ for all $s \in I$ then all nearby points k on $L \setminus D$ would have $k \in A_s$ for $s \in I$ and $k \in \overline{A_s}$ for $s \notin I$, so such points would lie in D which is impossible.

Therefore $h \in Z_s$ for some $s \in I$. Since $g \in A'_s$, we have $w(D) \subset \overline{A'_s}$ so $w(C) \subset A'_s$. By the lemma we have $\ell(sw) < \ell(w)$, so by induction applied to $h \in D$ and $sg \in sw(D)$ we get that the segment from h to sg has a finite cover in $\mathscr{C}$. Transforming this cover by s yields a finite cover of the segment from $sh = h$ to $s^2 g = g$ as needed. □