1 Last time: fundamental domain for W

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$.

Let $T = \{wsw^{-1} : w \in W,\ s \in S\}$.

Some things to recall:

Definition. The *Bruhat order* on W is the partial order $<$ which is the transitive closure of the relation with $w < wt$ whenever $w \in W$ and $t \in T$ and $\ell(wt) < \ell(w)$.

Note that if $s \in S$ and $w \in W$ then $ws < w$ if and only if $\ell(ws) < \ell(w)$.

Proposition. If $v, w \in W$ then $v \leq w$ if and only if for some (equivalently, every) reduced expression $w = s_1 \cdots s_r$ there are indices $1 \leq i_1 < i_2 < \cdots < i_q \leq r$ such that $v = s_{i_1} s_{i_2} \cdots s_{i_q}$.

Proposition. $(W, <)$ is a *graded* partially ordered set with *rank function* ℓ. In other words, every maximal chain in Bruhat order $v = w_0 < w_1 < \cdots < w_m = w$ has the same length $m = \ell(w) - \ell(v)$.

Assume S is a finite set.

Let $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$ be the usual W-module on which $s \in S$ acts by $sv = v - 2(\alpha_s, v)\alpha_s$ for $v \in V$, where $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is the bilinear form with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t))$.

Define V^* as the real vector space of $\mathbb{R}$-linear maps $V \to \mathbb{R}$. Since $|S| = n$ is finite, we can identify V and V^* with $\mathbb{R}^n$ and we give these spaces the standard Euclidean topology via this identification. The group W then acts on V^* as continuous linear transformations by the formula

$$(w\lambda)(v) = \lambda(w^{-1}v) \qquad \text{for } v \in V.$$

for $w \in W$ and $\lambda \in V^*$.

For each $s \in S$, define:

$$Z_s = \{f \in V^* : f(\alpha_s) = 0\} \qquad \text{and} \qquad A_s = \{f \in V^* : f(\alpha_s) > 0\}.$$

Next let

$$C = \bigcap_{s \in S} A_s = \{f \in V^* : f(\alpha_s) > 0 \text{ for all } s \in S\}.$$

Let D be the closure of C so that

$$D = \overline{C} = \{f \in V^* : f(\alpha_s) \geq 0 \text{ for all } s \in S\}.$$

This set has a partition given by the subsets

$$C_I = \left(\bigcap_{s \in I} Z_s\right) \cap \left(\bigcap_{s \notin I} A_s\right)$$

for $I \subset S$. The *Tits cone* is the set

$$U = \bigcup_{w \in W} w(D) = \bigcup_{w \in W} \bigcup_{I \subset S} w(C_I).$$

The main theorem from last time went as follows:

Theorem. The sets $\mathscr{C} = \{w(C_I) : w \in W,\ I \subset S\}$ form a partition of U, and W_I is the stabilizer of each point in C_I for $I \subset S$. Moreover, D is a *fundamental domain* for the W-action on U, meaning that each W-orbit in U intersects D in exactly one point. Finally, U is a convex cone.

2 Finiteness criteria

Our goal in the next few lectures is to outline the classification of the finite Coxeter groups (which will turn out to all be finite reflection groups). For this, we need to develop efficient methods detecting whether a given Coxeter graph generates a finite group.

Recall the notion of an *irreducible* Coxeter group: one whose Coxeter graph is connected. We proved the following statement earlier, under the hypothesis that W is a finite reflection group.

Proposition. Let (W, S) be a Coxeter system. Let $\Gamma_1, \dots, \Gamma_r$ be the connected components of the Coxeter graph of (W, S). Let $S_1, \dots, S_r$ be the sets of vertices in these components. Then $W = W_{S_1} \times \cdots \times W_{S_r}$ and each pair (W_{S_i}, S_i) is irreducible.

Proof. The same proof works as for reflection groups, now that we know that $W_I \cap W_J = W_{I \cap J}$. □

Corollary. The group W is finite if and only if $|S| < \infty$ and each irreducible component of (W, S) is a finite Coxeter system.

Thus we only need to determine the finite irreducible Coxeter groups.

For this, we relate finiteness to a topological condition on the geometric representation of W. As usual, let $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$ be the W-module with $s \in S$ acting by $sv = v - 2(\alpha_s, v)\alpha_s$ for $v \in V$, where $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is the bilinear form with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s, t))$.

Assume $|S| = n$ is finite and identify V with $\mathbb{R}^n$, and $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$, where $\mathbb{R}^{n\times n}$ is the vector space of $n \times n$ matrices over $\mathbb{R}$. Note that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ and for any $A \in \mathrm{GL}(n, \mathbb{R})$ the map $X \mapsto AX$ is a homeomorphism $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ (that is, a continuous map with a continuous inverse).

By passing to a dual basis, we may also identify (topologically) the dual space V^* with $\mathbb{R}^n$ and $\mathrm{GL}(V^*)$ with $\mathbb{R}^{n\times n}$.

Proposition. Let $f \in V^*$. The map $\mathrm{GL}(V^*) \to V^*$ given by $A \mapsto Af$ is continuous.

Proof. Write the map in coordinates: each of these is a linear function, which is continuous. □

Recall that $C = \{f \in V^* : f(\alpha_s) > 0 \text{ for all } s \in S\}$ is open.

Fix an element $f \in C$ and let C_0 be the inverse image of C under the map $\mathrm{GL}(V^*) \to V^*$ given by $A \mapsto Af$.

Proposition. The set C_0 is an open neighborhood of $1 \in \mathrm{GL}(V^*)$.

Proof. The set C_0 is the inverse image of an open set under a continuous map, and $1 \in C_0$ since $1f = f \in C$. □

Let $\sigma^* : W \to \mathrm{GL}(V^*)$ be the representation corresponding to the W-module structure we defined earlier on V^*, so that $\sigma^*(s) \in \mathrm{GL}(V^*)$ is the linear transformation with $(\sigma^*(s)\lambda)(v) = \lambda(sv)$ for $\lambda \in V^*$, $v \in V$, and $s \in S$.

Proposition. It holds that $\sigma^*(W) \cap C_0 = \{1\} \subset \mathrm{GL}(V^*)$.

Proof. Let $w \in W$. If $\sigma^*(w) \in C_0$ then $\sigma^*(w)(f) \in C$. But recall that $D \supset C$ is a fundamental domain for W, and contains only one element from each W-orbit. Hence, as $\sigma^*(1)(f) = f \in C$, every $\sigma^*(w) \in C_0$ must have $\sigma^*(w)(f) = f$, so $\sigma^*(W) \cap C_0$ must be contained in the image under σ^* of the pointwise stabilizer in W of $f \in C$. But we have see that this stabilizer is $W_\varnothing = \{1\}$ since $C = C_\varnothing$. □

Let $w_0 \in W$ and set $g = \sigma^*(w_0) \in \sigma^*(W) \subset \mathrm{GL}(V^*)$. The set gC_0 is then an open neighborhood of g.

Proposition. It holds that $gC_0 \cap \sigma^*(W) = \{g\}$.

Proof. If $\sigma_*(w) \in gC_0$ then $\sigma^*(w_0^{-1}w) \in C_0$ so $\sigma^*(w_0^{-1}w) = 1$ and hence $\sigma^*(w) = g$. □

A set A in a topological space X is *discrete* if for each $x \in X$ there exists an open set $U \subset X$ with $x \in U$ and $|A \cap U| \leq 1$. (Be aware that "discrete" is sometimes used to describe a slightly weaker condition: namely, that every $x \in A$ has an open neighborhood $U \subset X$ with $A \cap U = \{x\}$.)

For example, the set $\{\frac{1}{n} : n = 2, 3, 4, \dots\}$ is a discrete subset of $(0, 1)$ but not $[0, 1]$. (However, this set would be considered a discrete subset of $[0, 1]$ under the alternate definition mentioned.)

Lemma. It holds that $\sigma^*(W)$ is a discrete subset of $\mathrm{GL}(V^*)$.

Proof. For each point $g \in \mathrm{GL}(V^*)$, we need to produce an open neighborhood U of g with $|\sigma^*(W)\cap U| \leq 1$. If $g \in \sigma^*(W)$ then we can take $U = gC_0$. If $g \notin \sigma^*(W)$, then either $g \in hC_0$ for some $h \in \sigma^*(W)$ in which case we can take $U = hC_0$, or g must have an open neighborhood U disjoint from $\sigma^*(W)$: take U to be the intersection of $\mathrm{GL}(V^*)$ with an open ball centered at g of radius $\epsilon > 0$, where ϵ is such that $B_\epsilon \cap \mathrm{GL}(V^*) \subset C_0$ where B_ϵ is the open ball of radius ϵ centered at the origin. □

Let $\sigma : W \to \mathrm{GL}(V)$ be the representation corresponding to the W-module structure on V. Putting things together gives us the first main result of today:

Theorem. It holds that $\sigma(W)$ is a discrete subset of $\mathrm{GL}(V)$.

Proof. This follows from the lemma since when we identify $\mathrm{GL}(V)$ and $\mathrm{GL}(V^*)$ with $\mathbb{R}^{n\times n}$ as topological spaces, the transpose map affords a homeomorphism $\mathrm{GL}(V^*) \to \mathrm{GL}(V)$ mapping $\sigma^*(W) \to \sigma(W)$. □

Our second main theorem now goes as follows.

Theorem. If the bilinear form $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is positive definite then W is finite.

Proof. Assume that the form on V is positive definite. Then V is just a Euclidean space and we can identify $\sigma(W) \subset \mathrm{GL}(V)$ with a subgroup of the orthogonal group $O(n, \mathbb{R})$, whose element are the $n \times n$ invertible real matrices X with $X^{-1} = X^T$.

Lemma. The group $O(n, \mathbb{R})$ is a compact subset of $\mathbb{R}^{n\times n}$.

Proof. It suffices to show that $O(n, \mathbb{R})$ is closed and bounded. The set is closed since $X \in O(n, \mathbb{R})$ if and only if $(XX^T)_{ij} = \sum_{k=1}^n X_{ij}X_{jk} = \delta_{ij}$ for all $i, j \in [n]$, so the group is the zero locus of a finite number of polynomial equations. This also shows that $O(n, \mathbb{R})$ is bounded, since we have $\sum_{k=1}^n X_{ik}^2 = 1$ for all elements X is the group. □

By the previous theorem, $\sigma(W)$ is thus a discrete subgroup of a compact (Hausdorff) group.

Lemma. A discrete subset of a compact Hausdorff space is finite.

Proof. Let D be a discrete subset of a compact Hausdorff space K. For each $x \in K$, let U_x be an open neighborhood of x with $|U_x \cap D| \leq 1$. Since K is Hausdorff, if $x \notin D$ and there exists $y \in U_x \cap D$, then there are disjoint open sets V_x and V'_x with $x \in V_x$ and $y \in V'_x$. In this case, replace U_x by V_x. We then have an open cover $\{U_x\}_{x\in K}$ of K with the property that $U_x \cap D = \varnothing$ if $x \notin D$. By compactness, there exists a finite subcover $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ of K. Since every element of D belongs to exactly one of these sets, it follows that $|D| \leq n$. □

Combining these lemmas, we deduce that $\sigma(W)$ is finite. Since σ is injective, W is also finite. $\square$

It turns out that the converse of the preceding theorem is also true: the bilinear form associated to any finite Coxeter group is positive definite. To prove this, we will need to analyze the *radical* of the form $(\cdot,\cdot) : V \times V \to \mathbb{R}$, which is defined as the subspace

$$V^\perp = \{v \in V : (v, u) = 0 \text{ for all } u \in V\}.$$

Proposition. It holds that $V^\perp$ is a proper W-invariant subspace.

Proof. If $v \in V^\perp$ and $w \in W$ then $(wv, u) = (v, w^{-1}u) = 0$ for all $u \in V$. The set $V^\perp$ is clearly a subspace, and is not all of V since $\alpha_s \notin V^\perp$ for all $s \in S$. $\square$

Define $H_s = \{v \in V : (v, \alpha_s) = 0\}$ for $s \in S$.

Proposition. It holds that $V^\perp = \bigcap_{s\in S} H_s$.

Proof. Clearly we have $V^\perp \subset \bigcap_{s\in S} H_s$. If $(v, \alpha_s) = 0$ for all $s \in S$, then $(v, u) = 0$ for all $u \in V$ since $\{\alpha_s : s \in S\}$ is a basis of V. $\square$

Next time: more properties of $V^\perp$ and a proof that $(\cdot,\cdot)$ is positive definite if W is finite.