1 Finite Coxeter groups

Let (W, S) be a Coxeter system with $|S| < \infty$. Write $m(s,t)$ for the order of $st \in W$ for $s,t \in S$.

Define $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$ as the usual geometric representation of (W,S). Let $(\cdot,\cdot) : V \times V \to \mathbb{R}$ be the symmetric bilinear form with $(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t))$ for $s,t \in S$.

Theorem. If the bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is positive definite then W is finite. Moreover, with respect to the embedding $W \hookrightarrow \mathrm{GL}(V)$ given by the W-module structure on V, it holds that W is a discrete subset of $\mathrm{GL}(V)$.

Today: we will prove that the bilinear form associated to any finite Coxeter group is positive definite.

The *radical* of the form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is the subspace

$$V^\perp = \{v \in V : (v,u) = 0 \text{ for all } u \in V\}.$$

Note that $V^\perp$ is W-invariant, and $V^\perp = \bigcap_{s\in S} H_s$ where $H_s = \{v \in V : (v, \alpha_s) = 0\}$ for $s \in S$.

By definition, the form $(\cdot,\cdot)$ is *degenerate* if $V^\perp$ is nonzero.

Proposition. Assume (W,S) is irreducible, so that its Coxeter graph is connected.

(a) Every proper W-invariant subspace of V is contained $V^\perp$.

(b) If $(\cdot,\cdot)$ is degenerate then V is not a completely reducible W-module.

(c) If $(\cdot,\cdot)$ is nondegenerate then V is an irreducible W-module.

(d) The only endomorphisms of V commuting with the action of W are scalar maps.

Proof. We have seen that if $s \in S$ then s acts as -1 on the line $\mathbb{R}\alpha_s$ and fixes H_s pointwise, and $V = \mathbb{R}\alpha_s \oplus H_s$ Suppose V' is a W-invariant subspace of V. If $\alpha_s \notin V'$ for $s \in S$ then s must fix V' pointwise, so $V' \subset H_s$. Therefore if $\alpha_s \notin V'$ for all $s \in S$ then $V' \subset \bigcup_{s\in S} H_s = V^\perp$. If $\alpha_s \in V'$ for some $s \in S$, then any $t \in S$ with $m(s,t) > 2$ must have $\alpha_t \in V'$ since $\alpha_t = c(\alpha_s - t\alpha_s) \in V'$ for $c = \frac{1}{2(\alpha_s,\alpha_t)}$. In this case, since (W,S) is irreducible, it follows that $\alpha_t \in V'$ for all $t \in S$ so $V' = V$. This proves (a).

To prove (b), assume the form $(\cdot,\cdot)$ is degenerate. Then $V^\perp$ is a proper nonzero W-invariant subspace, but there cannot exist a complementary W-invariant subspace $U \subset V$ with $V = V^\perp \oplus U$ since every W-invariant subspace is contained in $V^\perp$. Therefore V is not completely reducible.

On the other hand, if $(\cdot,\cdot)$ is nondegenerate, then $V^\perp = 0$, so part (a) implies that V has no proper nonzero W-invariant subspaces, and is therefore irreducible as a W-module. This proves (c).

Finally, suppose $f : V \to V$ is a linear map with $f(wv) = wf(v)$ for all $w \in W$ and $v \in V$. Fix $s \in S$. We then have $-f(\alpha_s) = f(-\alpha_s) = f(s\alpha_s) = sf(\alpha_s)$, so $f(\alpha_s)$ is contained in the (-1)-eigenspace of s in V, which we is the line $\mathbb{R}\alpha_s$. Hence there exists $\lambda \in \mathbb{R}$ such that $f(\alpha_s) = \lambda\alpha_s$. Define

$$V' = \{v \in V : f(v) = \lambda v\}.$$

Then V' is a W-invariant subspace since $f(wv) = wf(v) = \lambda wv$ for $w \in W$ and $v \in V'$. As $\alpha_s \in V'$, we have $V' \not\subset V^\perp$, so $V' = V$ by part (a). This proves that $f(v) = \lambda v$ for all $v \in V$, which proves (d). $\square$

Next, we have a general lemma about group representations.

Lemma. Let $\rho : G \to \mathrm{GL}(E)$ be a group representation, with E a finite-dimensional real vector space.

(a) If G is finite then there exists a positive definite $\rho(G)$-invariant symmetric bilinear form on E.

(b) If G is finite then ρ is completely reducible.

(c) Suppose the only endomorphisms of E commuting with $\rho(G)$ are scalar maps. If β and β' are nondegenerate $\rho(G)$-invariant symmetric bilinear forms on E, then $\beta' = \lambda\beta$ for some $\lambda \in \mathbb{R}$.

Proof. Assume G is finite. Let β be any positive definite symmetric bilinear form on E. For example, fix a basis $e_1, e_2, \dots, e_n$ and then define $\beta(u, v) = \sum_{i=1}^n u_i v_i$ where $v = \sum_i v_i e_i$ and $u = \sum_i u_i e_i$. Let

$$\tilde{\beta}(u, v) = \sum_{g \in G} \beta(gu, gv).$$

This makes sense as G is finite, and you can check directly that $\tilde{\beta}$ is $\rho(G)$-invariant, symmetric, and positive definite. If $F \subset E$ is any subspace and $F^\perp = \{v \in E : \tilde{\beta}(v, f) = 0 \text{ for all } f \in F\}$ then $E = F \oplus F^\perp$ since $\tilde{\beta}$ is positive definite. Moreover, if F is $\rho(G)$-invariant then $F^\perp$ is as well. Hence E is completely reducible as a G-module. This proves (a) and (b).

To prove (c), suppose β and β' are nondegenerate $\rho(G)$-invariant symmetric bilinear forms on E. Then maps $E \to \mathrm{E}^*$ given by $Av = \beta(v, \cdot)$ and $Bv = \beta'(v, \dots)$ for $v \in E$ are then isomorphisms (this is what nondegenerate means), and $A^{-1}B : E \to E$ is an endomorphism commuting the action of G. (Check this!) By hypothesis, it follows that $A^{-1}B$ acts on E as some scalar $\lambda \in \mathbb{R}$, so $A^{-1}Bv = \lambda v$ and therefore $\beta'(v, w) = \beta(\lambda v, w) = \lambda\beta(v, w)$ for all $v, w \in V$. $\square$

Here is today's main theorem.

Theorem. Let (W, S) be a Coxeter system. The following are equivalent:

(a) W is finite.

(b) The bilinear form $(\cdot, \cdot)$ on the geometric representation V of W is positive definite.

(c) W is a finite reflection group.

Proof. We may assume that (W, S) is irreducible. We have already seen that (b) $\Rightarrow$ (c) $\Rightarrow$ (a). We need to show that (a) $\Rightarrow$ (b), so assume W is finite.

The result follows by combining the keys facts in the preceding proposition and lemma. By part (b) of the lemma, W acts completely reducibly on V, so $(\cdot, \cdot)$ must be nondegenerate by part (b) of the proposition. Therefore V is irreducible as a W-module, so the scalar maps are the only endomorphisms of V commuting with the action of W. This means that every nondegenerate symmetric W-invariant bilinear form on V is a scalar multiple of $(\cdot, \cdot)$. But there exists such a form which is positive definite by the lemma as W is finite. Since $(\alpha_s, \alpha_s) > 0$, it must hold that $(\cdot, \cdot)$ is a positive scalar multiple of this form; any positive multiple of a positive definite bilinear form is still positive definite. $\square$

2 Crystallographic Coxeter groups

This topic is sort of a digression, and most of the proofs in this section will only be sketches. There are two different common usages of the term "crystallographic Coxeter group" in the literature, which can often be confusing. With the language we've built up so far, we can now precisely explain the differences between these definitions.

A *lattice* in a finite-dimensional vector space V is a set of the form $L = \mathbb{Z}\text{-span}\{b_1, b_2, \dots, b_n\}$ where $b_1, b_2, \dots, b_n$ is a basis for V. A group W acting linearly on V stabilizes the lattice L if $wL = L$ for all $w \in W$. In this case, the trace of (the linear transformation $V \to V$ representing) $w \in W$ must be an integer, since its matrix in the basis $b_1, b_2, \dots, b_n$ has all integer entries. Here is the Coxeter theoretic definition of crystallographic:

Definition. A Coxeter group W is *crystallographic* (relative to its geometric representation) if W stabilizes some lattice L in V.

Lemma. If W is finite then W is crystallographic only if $m(s,t) \in \{1,2,3,4,6\}$ for all $s,t \in S$.

The converse of this lemma also holds, as we will see in a minute.

Proof. Assume W is finite and crystallographic. Then the trace of $\rho(w)$ is always an integer for $w \in W$. For any $s \neq t$ in S, the product st acts on V as a rotation by $2\pi/m(s,t)$ in the hyperplane spanned by α_s, β_t, while fixing pointwise the orthogonal complement of this subspace (since the associated bilinear form in the finite case is positive definite). Therefore the trace of st is $(|S| - 2 + 2\cos(2\pi/m(s,t))$ which is an integer if and only if $m(s,t) \in \{2,3,4,6\}$. □

Lemma. If W is any Coxeter group which is crystallographic then $m(s,t) \in \{1,2,3,4,6,\infty\}$ for $s,t \in S$.

Proof. When $s,t \in S$ are such that $m(s,t) \neq \infty$, the same argument applies as in the previous proof. □

In Lie theory, the condition in the preceding lemma is often taken as the definition of a crystallographic Weyl/Coxeter group. In the finite case, this turns out to be equivalent to our definition, but in the infinite case gives a slightly more general class of groups by the following theorem.

Theorem. A Coxeter group W is crystallographic if and only if (1) $m(s,t) \in \{1,2,3,4,6,\infty\}$ for all $s,t \in S$, and (2) in each cycle in the Coxeter graph of W the number of edges labeled 4 is even, and the number of edges labeled 6 is even.

Proof sketch. First assume W is crystallographic. Suppose we have a cycle $s_1 \to s_2 \to s_3 \to \cdots \to s_r \to s_1$ in the Coxeter graph of W and $m_{ij} = m(s_i,s_j)$. Try computing the trace of $s_1 s_2 \cdots s_r$. Deduce that this is in $\mathbb{Z}$ only if $\cos(\frac{\pi}{m_{12}})\cos(\frac{\pi}{m_{23}})\cdots\cos(\frac{\pi}{m_{r1}}) \in \mathbb{Q}$, which holds only if the number of labels among $m_{12,}, m_{23}, \ldots, m_{r1}$ (which are already each either 3, 4, 6, or ∞) which are 4 (respectively, 6) is even.

Conversely, suppose W satisfying the given conditions. To show that W is crystallographic, one must exhibit a lattice $L \subset V$ stabilized by W. The lattice $L = \mathbb{Z}\text{-span}\{\alpha_s : s \in S\}$ doesn't have this property, but we can construct a W-stabilized lattice of the form $L = \mathbb{Z}\text{-span}\{\lambda_s : s \in S\}$ where each $\lambda_s = c_s\alpha_s$ for some $c_s \in \mathbb{R}$. In particular, one checks that L is stabilized by W whenever the coefficients c_s satisfy

$$c_s = c_t \text{ if } m(s,t) \text{ is 3 or } \infty,$$
$$c_s = \sqrt{2}c_t \text{ or } c_t = \sqrt{2}c_s \text{ if } m(s,t) = 4.$$
$$c_s = \sqrt{3}c_t \text{ or } c_t = \sqrt{3}c_s \text{ if } m(s,t) = 6,$$

The work left to be done is to show that there exist a choice of real numbers $c_s \in \mathbb{R}$ satisfying these conditions. The cycle conditions ensure that we can always find such scalars. (Try this yourself!) □

3 Coxeter graphs of positive type

We can say which bilinear forms on V correspond to finite groups W: the positive definite ones. How do we detect this condition from the Coxeter graph of W? In other words, which Coxeter graphs give rise to finite Coxeter groups?

We will sketch the classification of these graphs next time. The key idea in the proof of the classification is the following lemma. Say that a Coxeter graph Γ is a *positive type* if the bilinear form of the geometric representation of the associated Coxeter group if *positive semidefinite*, meaning that $(v,v) \geq 0$ for all vectors v. Likewise, Γ is *positive definite* if this form is positive definite.

Lemma. If Γ is of positive type then every proper subgraph of Γ is positive definite.

Using this fact, we can classify all Coxeter graphs of positive type. It is then not too hard to identify which of these graphs is positive definite. We'll look at this next time.