1 Last time: finiteness criteria

Recall the main theorem from last time:

Theorem. Let (W, S) be a Coxeter system. Assume S is finite. The following are equivalent:

(a) W is finite.

(b) The bilinear form $(\cdot, \cdot)$ on the geometric representation V of W is positive definite.

(c) W is a finite reflection group.

We also briefly discussed crystallographic groups.

A *lattice* in a finite-dimensional vector space V is a set of the form $L = \mathbb{Z}\text{-span}\{b_1, b_2, \dots, b_n\}$ where $b_1, b_2, \dots, b_n$ is a basis for V. A Coxeter group W is *crystallographic* (relative to its geometric representation) if there exists a lattice $L \subset V$ with $wL = L$ for all $w \in W$.

Theorem. A Coxeter group W is crystallographic if and only if $m(s,t) \in \{1, 2, 3, 4, 6, \infty\}$ for all $s, t \in S$, and in each cycle in the Coxeter graph of W the number of edges labeled 4 is even, and the number of edges labeled 6 is even.

Sometimes in the literature, the cycle condition is ignored and W is said to be crystallographic if $m(s,t) \in \{1, 2, 3, 4, 6, \infty\}$ for all $s, t \in S$. The two definitions are equivalent if W is finite, as we will see later today.

2 Classification of finite Coxeter groups

Today, we will sketch the proof of the complete classification of the finite and affine Coxeter systems. Some steps in this proof are presented as exercises on this week's homework assignment.

The first theorem shows that to classify the finite Coxeter groups, it suffices to identify the connected Coxeter graphs Γ (with a finite number of vertices) whose associated bilinear form, defined by

$$(\alpha_s, \alpha_t) = -\cos(\pi/m) \qquad \text{for vertices } s, t \text{ in } \Gamma \text{ connected by an edge with label } m,$$

is postive definite, i.e., has $(v, v) > 0$ for all $0 \neq v \in V = \mathbb{R}\text{-span}\{\alpha_s : s \text{ is a vertex of } \Gamma\}$.

Say that a Coxeter graph Γ is a *positive type* if its associated bilinear form if *positive semidefinite*, meaning that $(v, v) \geq 0$ for all vectors v. Say that Γ is *positive definite* if the associated form is positive definite.

Suppose the vertices of Γ are $S = \{s_1, s_2, \dots, s_n\}$. Let $\alpha_i = \alpha_{s_i}$ for $i \in [n]$.

The matrix of the bilinear form associated to Γ is then

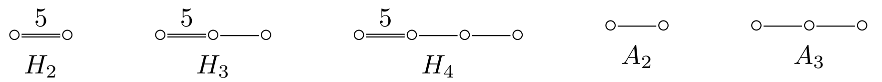
$$M = \begin{bmatrix} (\alpha_1, \alpha_1) & \dots & (\alpha_1, \alpha_n) \\ \vdots & & \vdots \\ (\alpha_n, \alpha_1) & \dots & (\alpha_n, \alpha_n) \end{bmatrix}.$$

The *principal minors* of M are the determinants of the submatrices given by removing a commensurate set of k rows and columns. (If we remove row i then we must also remove column i, and so on.)

Proposition. The bilinear form associated to a Coxeter graph Γ is positive definite (respectively, semidefinite) if and only if all principal minors of the corresponding matrix M are positive (nonnegative).

Proof. This is a standard fact from linear algebra, which we won't prove in detail. To get an idea of why this holds, consider the case when M is diagonal. Then all principal minors are positive/nonnegative if and only if all diagonal entries are positive/nonnegative. To deduce the general case, use the fact that M can be diagonalized. □

Example. Consider the Coxeter graphs



Let M_Γ be the matrix of the bilinear form associated to Coxeter graph Γ of these types. We then have

$$2M_{A_2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$2M_{A_3} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$2M_{H_2} = \begin{bmatrix} 2 & -c \\ -c & 2 \end{bmatrix} \qquad \text{for } c = 2\cos(\pi/5) = \frac{1+\sqrt{5}}{2}.$$

$$2M_{H_3} = \begin{bmatrix} 2 & -c & 0 \\ -c & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

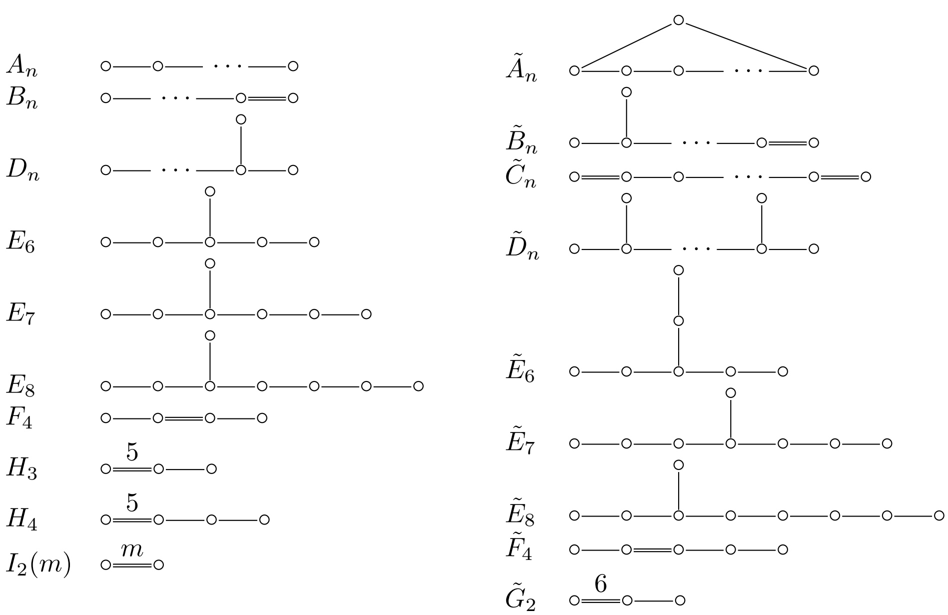
$$2M_{H_4} = \begin{bmatrix} 2 & -c & 0 & 0 \\ -c & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

We have multiplied these matrices by 2 to make the numbers a little nicer. Note that this just rescales the corresponding principal minors by a power of 2, so has no effect on whether those numbers are positive or nonnegative.

Anyways, by examining these matrices, we see that every principal minor or M_{H_4} is the determinant of 1, M_{A_2}, M_{A_3}, M_{H_2}, M_{H_3}, or M_{H_4}. All such determinants are positive (try computing this!), so the Coxeter graph of type H_4 is positive definite.

The following is one half of the classification theorem:

Theorem. The following Coxeter graphs have positive type:



Among, these, only the graphs in the left column are positive definite.

(Here, recall that unlabeled edges have weight 3, and unlabeled doubled edges have weight 4.)

Proof sketch. Proceed as in the example for H_4. Note that in graph X_n the subscript gives the number of vertices; call this the *rank*. Observe that in each graph, removing any number of vertices gives a disjoint union of graphs of smaller rank. Hence all principle minors of a given graph are products of determinants of the matrices M_Γ of the bilinear forms associated to graphs Γ of smaller rank. One can check directly the requisite positivity/nonnegativity for the exceptional graphs of type E, F, G, H, I. For the infinite families of type A, B, C, and D, use the preceding observations to find an inductive formula for $\det M_\Gamma$: this exercise is part of this week's homework. □

The second half of the classification is the following theorem which we'll spend the rest of today proving:

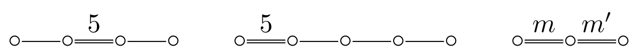
Theorem. The graphs in the previous theorem are the only connected Coxeter graphs of positive type.

To prove this, we will apply two lemmas as black boxes.

Lemma. If Γ is a connected Coxeter graph of positive type then every proper subgraph of Γ is positive definite.

Proof. You will prove this in the homework! □

Lemma. The Coxeter graphs



are not of positive type if $m = 3$ and $m' \geq 7$, or if $m \geq 4$ and $m' \geq 5$.

Proof. Check this by direct calculation. In the last case, for example, the determinant of the matrix of the associated bilinear form is $1 - \cos^2(\pi/m) - \cos^2(\pi/m')$, which is negative is the proscribed cases. □

Proof of theorem. Let Γ be a connected Coxeter graph with n vertices, and maximum edge label m. Assume Γ has positive type but is not given by one of the given graphs. We try to deduce a contradiction.

(1) All graphs of ranks 1 or 2 are of type A_1, $I_2(m)$, or $\tilde{A}_1$, so we must have $n \geq 3$.

(2) $\tilde{A}_1$ cannot be a subgraph of Γ since every proper subgraph of Γ is positive definite, and Γ is not equal to $\tilde{A}_1$ either. Therefore $m < \infty$.

(3) By the second lemma, and since $\tilde{A}_n$ for $n \geq 2$ cannot be a subgraph, it follows that Γ has no cycles.

Suppose $m = 3$.

(4) Γ must have a branch point since $\Gamma \neq A_n$.

(5) Since Γ does not have $\tilde{D}_n$ as a subgraph, there is only one branch point.

(6) Since Γ does not have $\tilde{D}_4$ as a subgraph, the branch point can only have three branches. Suppose these have lengths $a \leq b \leq c$ so that $n = 1 + a + b + c$.

(7) Since $\tilde{E}_6$ is not a subgraph, $a = 1$.

(8) Since $\tilde{E}_7$ is not a subgraph, $b \leq 2$.

(9) Since $\Gamma \neq D_n$, we cannot have $b = 1$ so $n = 2$.

(10) Since $\tilde{E}_8$ is not a subgraph, $c \leq 4$

But this implies that Γ is either E_6, E_7, or E_8, contradicting our hypothesis. Thus we cannot have $m = 3$. Suppose instead that $m = 4$:

(12) Since Γ does not have $\tilde{C}_n$ as a subgraph, there is only one edge labeled 4.

(13) Since $\tilde{B}_n$ is not a subgraph, there are no branch points.

(14) Since $\Gamma \neq B_n$, the single edge labeled 4 cannot be an extreme edge.

(15) Since $\tilde{F}_4$ is not a subgraph, we must have $n = 4$.

But this implies that Γ is F_4, again a contradiction. Thus we cannot have $m = 4$. Finally suppose $m \geq 5$.

(16) By the second lemma, there can only be one edge with label > 3, and this label must be m.

(17) Since $\tilde{G}_2$ is not a subgraph and since Γ is not $I_2(m)$, we cannot have $m = 6$. Likewise, since none of the graphs in the second lemma are subgraphs, we cannot have $m \geq 7$. Therefore $m = 5$.

(18) Since Γ does not contain the first two graphs in the second lemma as subgraphs, the unique edge labeled 5 must be extreme, and n must be 3 or 4.

But this implies that Γ is H_3 or H_4, again a contradiction. This eliminates all possibilities for Γ, and so completes our proof of theorem. □

This finishes the classification of the finite Coxeter groups, which also give all finite reflection groups! Next time, we'll begin the second major theme of the course, introducing the Iwahori-Hecke algebra of a Coxeter system.