1 Last time: generic Hecke algebra

Let (W, S) be a Coxeter system.

Let A be a commutative ring with unit 1.

Let $\mathcal{H} = \mathcal{H}_{A,W}$ be the free A-module with basis $\{T_w : w \in W\}$.

Choose elements $a_s, b_s \in A$ for $s \in S$ such that $a_s = a_t$ and $b_s = b_t$ if $s, t \in S$ are conjugate in W.

Last time, we prove this fundamental result:

Theorem. There is a unique associative A-algebra structure on $\mathcal{H}$ with unit T_1 and such that

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ a_s T_w + b_s T_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases} \qquad \text{if } s \in S \text{ and } w \in W. \tag{*}$$

We call this algebra a *generic (Hecke) algebra* of (W, S).

Remark. Recall that an A-algebra M is an A-module with an associative, A-bilinear multiplication $M \times M \to M$ and a unit element 1 such that $1x = x1 = x$ for all $x \in M$. This is a very mild generalization of a ring, which is itself just a $\mathbb{Z}$-algebra. The hard part of the preceding theorem was showing that the product defined by (*) is associative.

Some important facts about $\mathcal{H}$:

1. In the generic algebra, it also holds that

$$T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w) \\ a_s T_w + b_s T_{ws} & \text{if } \ell(ws) < \ell(w) \end{cases} \qquad \text{if } s \in S \text{ and } w \in W.$$

2. If $w = s_1 s_2 \cdots s_k$ is any reduced expression for $w \in W$, then $T_w = T_{s_1} T_{s_2} \cdots T_{s_n}$.

2 Presentation for $\mathcal{H}$

Today's main result will be to prove that $\mathcal{H}$ can alternatively be defined as the algebra generated by the set $\{T_s : s \in S\}$ subject to relations which are similar to the Coxeter relations defining the group W. For this, we first need a few technical properties related to the Bruhat order.

Write $<$ the Bruhat order on W. Recall:

Lifting property. If $x, y \in W$ and $s \in S$ then

$$x \leq y \qquad \Rightarrow \qquad (xs \leq y \text{ or } xs \leq ys) \text{ and } (sx \leq y \text{ or } sx \leq sy).$$

We also note this corollary of the main theorem from last time:

Corollary. There exists a unique associative product $\circ : W \times W \to W$ with

$$s \circ w = \begin{cases} sw & \text{if } sw < w \\ w & \text{else} \end{cases} \qquad \text{and} \qquad w \circ s = \begin{cases} ws & \text{if } ws < w \\ w & \text{else} \end{cases}$$

for $s \in S$ and $w \in W$.

Proof. This describes multiplication (of the basis elements T_w) in $\mathcal{H}$ when $a_s = 1$ and $b_s = 0$. □

Lemma. Let $s_1, \dots, s_n \in S$ and set $w = s_1 \circ \cdots \circ s_n \in W$. Then $T_{s_1} T_{s_2} \cdots T_{s_n} \in a(s_1, \dots, s_n) T_w + A\text{-span}\{T_v : v < w\}$ for a constant $a(s_1, \dots, s_n) \in A$.

Proof. This is clear if $n = 0$: then the product if 1 and we can take $a() = 1$. Suppose $n > 0$ and let $w' = s_2 \circ \cdots \circ s_n$. Suppose $T_{s_2} \cdots T_{s_n} \in a(s_2, \dots, a_n) T_{w'} + A\text{-span}\{T_v : v < w'\}$.

Suppose $s_1 w' = w > w'$ so that $T_{s_1} T_{w'} = T_w$. Set $a(a_1, s_2, \dots, a_n) = a(s_2, \dots, a_n)$. It follows by the lifting property that $v < w$ and $sw < w$ if $v < w'$. Therefore

$$\begin{aligned} T_{s_1} T_{s_2} \dots T_{s_n} &\in T_{s_1} \left(a(s_2, \dots, a_n) T_{w'} + A\text{-span}\{T_v : v < w'\} \right) \\ &\subset a(s_1, s_2, \dots, a_n) T_w + A\text{-span}\{T_v : v < w\}. \end{aligned}$$

Suppose instead that $s_1 w' < w' = w$ so that $T_{s_1} T_{w'} = a_{s_1} T_w + b_{s_1} T_{sw}$. It follows again by the lifting property that if $v < w = w'$ then $sv \le w$. Therefore

$$\begin{aligned} T_{s_1} T_{s_2} \dots T_{s_n} &\in a_{s_1} a(s_2, \dots, a_n) T_w + b_{s_1} a(s_2, \dots, a_n) T_{sw} + A\text{-span}\{T_v : v \le w\} \\ &\subset a(s_1, s_2, \dots, a_n) T_w + A\text{-span}\{T_v : v < w\}. \end{aligned}$$

for some constant $a(s_1, s_2, \dots, a_n) \in A$. □

Let $\mathcal{F}$ be the free A-algebra on the set $\{F_s : s \in S\}$, so that $\mathcal{F}$ is the free A-module with a basis given by the symbols $F_{s_1} F_{s_2} \cdots F_{s_n}$ for all finite tuples $(s_1, s_2, \dots, s_n)$ with $s_i \in S$ and $n \in \mathbb{N}$, with multiplication of basis elements given by concatenation.

Fix an arbitrary total order $\prec$ on S. Define $F_w \in \mathcal{F}$ for $w \in W$ as the basis element $F_w = F_{s_1} F_{s_2} \cdots F_{s_n}$ where $w = s_1 s_2 \cdots s_n$ is the lexicographically minimal reduced expression for w relative to the order $\prec$ on S. In this notation, we have $F_1 = 1 \in \mathcal{F}$. Note that if $s_1 s_2 \dots s_n = t_1 t_2 \dots t_n$ in W (with $s_i, t_i \in S$), then $F_{s_1} F_{s_2} \cdots F_{s_n} = F_{t_1} F_{t_2} \cdots F_{t_n}$ in $\mathcal{F}$ if and only if $s_i = t_i$ for $i \in [n]$.

Now let $I \subset \mathcal{F}$ be the (two-sided) ideal generated by the relations

$F_s^2 = a_s F_s + b_s F_1$ for $s \in S$.

$F_s F_t F_s \cdots = F_t F_s F_t \cdots$ for $s, t \in S$, where both sides have $m(s, t)$ terms.

In other words, let I be the intersection of all ideals in $\mathcal{F}$ which contain the elements

$$a_s F_s + b_s F_1 - F_s \qquad \text{and} \qquad \underbrace{F_s F_t F_s \cdots}_{m(s,t) \text{ terms}} - \underbrace{F_t F_s F_t \cdots}_{m(s,t) \text{ terms}}$$

for all $s, t \in S$.

Write $f + I$ for the coset $\{f + x : x \in I\} \subset \mathcal{F}$.

Lemma. For any reduced expression $w = s_1 s_2 \cdots s_n \in W$, it holds that $F_{s_1} F_{s_2} \cdots F_{s_n} + I = F_w + I$.

Proof. This follows from the homework exercise in which you showed that any two reduced words can be transformed to each other by a sequence of braid moves. □

Lemma. For $s \in S$ and $w \in W$ it holds that $F_s F_w + I = \begin{cases} F_{sw} + I & \text{if } sw > w \\ a_s F_w + b_s F_{sw} + I & \text{else.} \end{cases}$

Proof. Write $F_w = F_{s_1} \cdots F_{s_n}$. If $sw > w$ then $sw = s s_1 \cdots s_n$ is also a reduced expression so $F_s F_w + I = F_{sw} + I$ by the previous lemma. If $sw < w$ then w has a reduced expression $w = s t_1 \cdots t_n$ so by the previous lemma $F_s F_w + I = F_s^2 F_{t_1} \cdots F_{t_n} + I = (a_s F_s + b_s F_1) F_{t_1} \cdots F_{t_n} + I = a_s F_w + b_s F_{sw} + I$. □

Corollary. If $s_1, \dots, s_n \in S$ then $F_{s_1} \cdots F_{s_n} + I = a(s_1, \dots, s_n) F_{s_1 \circ \dots \circ s_n} + A\text{-span}\{F_v : v < s_1 \circ \dots \circ s_n\}$, where $a(s_1, \dots, s_n) \in A$ is the same coefficient as in our earlier lemma.

Proof. This follows by induction from the preceding lemma. □

The universal property of a free algebra asserts that there is a unique surjective A-algebra homomorphism

$$\phi : \mathcal{F} \to \mathcal{H}$$

with $\phi(F_s) = T_s$ for all $s \in S$. It automatically holds that $\phi(F_w) = T_w$ for $w \in W$.

Clearly $I \subset \ker \phi$.

Proposition. $I = \ker \phi$.

Proof. Suppose $x \in \ker \phi$. Write $x = \sum_{(s_1, s_2, \dots, s_n) \in Z} b(s_1, s_n \dots, s_n) F_{s_1} F_{s_2} \cdots F_{s_n}$ for some set of tuples Z of elements of S and some coefficients $b(—) \in A$. It follows the lemmas above that we can write

$$x + I \in \sum_{w \in W} \left(\left(\sum_{\substack{(s_1, s_2, \dots, s_n) \in Z \\ s_1 \circ s_2 \circ \dots \circ s_n = w}} a(s_1, s_2, \dots, s_n) b(s_1, s_2, \dots, s_n) \right) F_w + A\text{-span}\{F_v : v < w\} \right) + I.$$

If the coefficient

$$c = \sum_{\substack{(s_1, s_2, \dots, s_n) \in Z \\ s_1 \circ s_2 \circ \dots \circ s_n = w}} a(s_1, s_2, \dots, s_n) b(s_1, s_2, \dots, s_n)$$

is nonzero for any $w \in W$, then whenever w is maximal in the Bruhat order of W such that $c \neq 0$, it holds that

$$\phi(x + I) \in cT_w + A\text{-span}\{T_v : v \neq w\}.$$

But this set does not contain 0, contradicting our assumption that $x \in \ker \phi$. Hence every such coefficient c must be zero, so $x \in I$. □

Putting things together, we conclude that:

Theorem. The map $\phi : \mathcal{F} \to \mathcal{H}$ has kernel I, so descends to an algebra isomorphism $\mathcal{F}/I \xrightarrow{\sim} \mathcal{H}$.

Equivalently, $\mathcal{H}$ is isomorphic to the A-algebra generated by T_s for $s \in S$, subject to the relations

(i) $T_s^2 = a_s T_s + b_s T_1$ for $s \in S$.

(ii) $T_s T_t T_s \cdots = T_t T_s T_t \cdots$ for $s, t \in S$, where both sides have $m(s,t)$ terms.

Note that (i) and (ii) become the relations defining the group W when $a_s = 0$ and $b_s = 1$.

Corollary. If $\mathcal{X}$ is an A-algebra and $\varphi : \{T_s : s \in S\} \to \mathcal{X}$ is map, then φ extends to a (unique) A-algebra homomorphism $\mathcal{H} \to \mathcal{X}$ if and only if the relations (i) and (ii) still hold with T_s and T_t replaced by their images under φ.

Proof. This is essentially the definition of what it means to say that $\mathcal{H}$ is generated by T_s for $s \in S$ subject to (i) and (ii). □

Corollary. For each $s \in S$, let $\theta_s \in A$ be a root of the equation $x^2 = a_s x + b_s$, and choose these roots such that $\theta_s = \theta_t$ if $s, t \in S$ are W-conjugate. Then there exists a unique A-algebra homomorphism $\mathcal{H} \to A$ with $T_s \mapsto \theta_s$ for $s \in S$.

Proof. For $s,t \in S$, we have $\theta_s^2 + a_s\theta_s + b_s$ by construction, and it holds that $\theta_s\theta_t \cdots = \theta_t\theta_s \cdots$ (both sides with $m(s,t)$ terms) since either $m(s,t)$ is even (so both sides are $(\theta_s\theta_t)^{m(s,t)/2}$) or $\theta_s = \theta_s$ since $m(s,t)$ is odd and s,t are conjugate in W. Thus relations (i) and (ii) hold for the map under consideration, so the result follows by the preceding corollary. □