1 Hecke algebras so far

Here's what we know about Hecke algebras so far.

Let (W, S) be a Coxeter system.

Let A be a commutative ring with unit 1.

Choose elements $a_s, b_s \in A$ for $s \in S$ such that $a_s = a_t$ and $b_s = b_t$ if $s, t \in S$ are conjugate in W.

The following definition is a little different from the one we've been using in past lectures, but is equivalent by the main result last time:

Definition. The *generic (Hecke) algebra* $\mathcal{H}$ is the (unital, associative) A-algebra generated by $\{T_s : s \in S\}$ subject to the relations

(1) $T_s^2 = a_s T_s + b_s T_1$ for $s \in S$.

(2) $T_s T_t T_s \cdots = T_t T_s T_t \cdots$ for $s, t \in S$, where both sides have $m(s, t)$ terms.

For $w \in W$ define $T_w \in H$ by $T_w = T_{s_1} T_{s_2} \cdots T_{s_k}$ where $w = s_1 s_2 \cdots s_k$ is any reduced expression. We get the same element in $\mathcal{H}$ for any choice of reduced expression as a result of relation (1). For $w = 1$, define T_1 as the unit in $\mathcal{H}$.

Inverting the order of results last time, we get:

Theorem. The algebra $\mathcal{H}$ is a free A-module with basis $\{T_w : w \in W\}$ and unit T_1, and it holds that

1. $T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ a_s T_w + b_s T_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}$ if $s \in S$ and $w \in W$.

2. $T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w) \\ a_s T_w + b_s T_{ws} & \text{if } \ell(ws) < \ell(w) \end{cases}$ if $s \in S$ and $w \in W$.

3. $T_x T_y = T_{xy}$ if $x, y \in W$ and $\ell(xy) = \ell(x) + \ell(y)$ and $T_s^2 = a_s T_s + b_s T_1$ for $s \in S$.

Moreover, $\mathcal{H}$ is the only A-algebra structure on the free A-module with basis $\{T_w : w \in W\}$ and unit T_1 satisfying any one of these properties.

The advantage of thinking of $\mathcal{H}$ via its presentation is:

Corollary. If $\mathcal{X}$ is an A-algebra and $\varphi : \{T_s : s \in S\} \to \mathcal{X}$ is map, then φ extends to a (unique) A-algebra homomorphism $\mathcal{H} \to \mathcal{X}$ if and only both of the following hold:

(1) $\phi(T_s)^2 = a_s \phi(T_s) + b_s 1_{\mathcal{X}}$ for $s \in S$.

(2) $\phi(T_s)\phi(T_t)\phi(T_s) \cdots = \phi(T_t)\phi(T_s)\phi(T_t) \cdots$ for $s, t \in S$, where both sides have $m(s, t)$ terms.

2 Hecke algebras from idempotents

The goal, before proceeding to more technical properties of $\mathcal{H}$, is to motivate our definition of the generic Hecke algebra of a Coxeter system as a special case of a much more general construction in representation theory. We'll realize half of this goal today, by setting up the main parts of the general theory of Hecke algebras. The rest will come next time.

We'll need a little commutative algebra. To make everything a bit simpler, we restrict our attention to the following well-behaved situation: let A be a finite-dimensional algebra over a field K. All algebras are associative and unital.

The *(Jacobson) radical* of A is the set of elements $J(A)$ consisting of the intersection of the annihilators of all simple left A-modules. Note that this set is actually a two-sided ideal.

The seeming preference for the left is not needed: one can show that the same definition using right A-modules gives the same thing. But we won't pursue those results here.

The algebra A is *(Jacobson) semisimple* if $J(A) = 0$.

Theorem (Wedderburn Theorem)**.** If A is semisimple then there are numbers $d_1, \dots, d_n \geq 1$ such that

$$A \cong \bigoplus_{i=1}^{n} K^{d_i \times d_i} \tag{*}$$

as K-algebras. Here $K^{d\times d}$ is the algebra of $d \times d$ matrices over K. So we can think of A, when semsimple, as an algebra of square, block diagonal matrices over K.

This is a fundamental result which we won't prove. This shows that the representation theory of a semisimple algebra is essentially trivial: all simple modules of $K^{d\times d}$ are isomorphic to K^d, and all simple modules of A will be direct sums of the simple modules of the factor algebras $K^{d_i\times d_i}$. However, usually the algebra A is given as a set of symmetries of some object, and it is highly nontrivial to interpret the decomposition of any given symmetric according to the isomorphism (*), or more generally to determine the dimensions d_i.

We prove two standard facts to make it easier to detect semisimplicity.

Proposition. If M is a finite-dimensional left A-module and $J(A)M = M$ then $M = 0$.

Proof. Assume $M \neq 0$ and let $J = J(A)$. Suppose $JM = M$. If M were simple then $JM = 0 \neq M$. Since M is finite-dimensional and not simple, there must exist a proper submodule $N \subset M$ such that M/N is simple: take N to be any proper submodule of maximal possible dimension. But then J annihilates M/N, so $JM \subset N \neq M$, contradicting our hypothesis. Therefore $M = 0$. □

If I is an ideal, then I^m is the ideal generated by all products $a_1a_2\cdots a_m$ with $a_i \in I$.

An ideal I is *nilpotent* if $I \neq 0$ and $I^m = 0$ for some $m \geq 1$.

Proposition. The radical $J(A)$ is a nilpotent ideal.

Proof. Let $J = J(A)$ Since A is finite-dimensional, the sequence $J \supset J^2 \supset J^3 \supset \dots$ must stabilize. But if $JJ^m = J^{m+1} = J^m$ for some m then by the previous proposition $J^m = 0$. □

Corollary. If A has no nilpotent left ideals (respectively, right ideals), then A is semisimple.

Proof. If A is not semisimple, then $J(A)$ is nilpotent two-sided ideal. □

Let $e \in A$ be an *idempotent*, that is, an element with $e^2 = e$.

Consider the right A-module $eA = \{ea : a \in A\}$.

Let $\mathrm{End}_A(eA)$ be the set of linear maps $\lambda eA \to eA$ with $\lambda(xa) = \lambda(x)a$ for $x \in eA$ and $a \in A$.

Let $\mathcal{H}(A, e) = eAe = \{eae : a \in A\}$. Call $\mathcal{H}(A, e)$ the *Hecke algebra of* (A, e).

Note that both $\mathrm{End}_A(eA)$ and $\mathcal{H}(A, e)$ are K-algebras: the product for the endomorphism algebra is composition, and the unit for the Hecke algebra is e. Therefore $eu = ue = eue$ for all $u \in \mathcal{H}(A, e)$.

The reason we introduce these algebras together is:

Proposition. The map $\phi(\lambda) = \lambda(e)$ is an algebra isomorphism $\mathrm{End}_A(eA) \to \mathcal{H}(A, e)$.

Proof. Clearly ϕ is linear and $\phi(1) = e$. If $\phi(\lambda) = eXe$ and $\phi(\lambda') = eYe$ for $X, Y \in A$ then

$$\phi(\lambda \circ \lambda') = \lambda(\lambda'(e)) = \lambda(eYe) = \lambda(e)Ye = eXeYe = \phi(\lambda)\phi(\lambda')$$

since $e^2 = e$. Thus ϕ is a homomorphism. We have $\phi(\lambda) = \lambda(e) = 0$ only if $\lambda(ea) = \lambda(e)a = 0$ for all $a \in A$, so ϕ is injective. Finally, ϕ is surjective since for any $X \in A$ we have $eXe = \phi(\lambda)$ for the endomorphism $\lambda \in \mathrm{End}_A(eA)$ with $\lambda(ea) = eXea$ for $a \in A$. □

Schur's lemma states that the right module eA is simple if and only if $\mathrm{End}_A(eA)$ is a division algebra.

The idempotent $e \in A$ is *primitive* if this condition holds and eA is simple.

Proposition. Suppose $u \in \mathcal{H}(A, e)$ is a primitive idempotent, so that $u\mathcal{H}(A, e)$ is a simple right $\mathcal{H}(A, e)$-module. Then u is also a primitive idempotent in A,and uA is simple submodule of eA.

Proof. Write $\mathcal{H} = \mathcal{H}(A, e)$ and let $u \in \mathcal{H}$ be an idempotent. Note that $u = eu = ue$. Using the previous proposition, we have $\mathrm{End}_A(uA) \cong uAu = ueAeu = u\mathcal{H}u \cong \mathrm{End}_{\mathcal{H}}(u\mathcal{H})$. It follows by Schur's lemma that u is primitive in $\mathcal{H}$ if and only if u is primitive in A. □

Proposition. If $M \subset eA$ is a simple A-module then $Me \subset \mathcal{H}(A, e)$ is a simple $\mathcal{H}(A, e)$-module or 0.

Proof. Let $M \subset eA$ be a simple A-module. If $x \in M$ and $xe \in Me \subset M$ is nonzero, then $xeA = M$ so $xe\mathcal{H}(A, e) = xeAe = Me$, so Me is a simple $\mathcal{H}(A, e)$-submodule of $\mathcal{H}(A, e)$. □

3 Hecke algebras from group algebras

We now specialize to the case when A is a group algebra and e is the idempotent affording the trivial representation of a subgroup. In this nice situation, both A and $\mathcal{H}(A, e)$ will be semisimple.

Let G be a finite group and let $B \subset G$ be a subgroup.

Consider the complex group algebra $\mathbb{C}G = \mathbb{C}\text{-span}\{g : g \in G\}$.

Let $e = \frac{1}{|B|}\sum_{b\in B} b \in \mathbb{C}G$.

Proposition. If $b \in B$ then $be = eb = e$ so $e^2 = e$.

Proof. An easy calculation. □

Define $\mathcal{H}(G, B) = \mathcal{H}(\mathbb{C}G, e) = e\mathbb{C}Ge = \mathbb{C}\text{-span}\{ege : g \in G\}$.

Abbreviate by writing $\mathcal{H} = \mathcal{H}(G, B)$.

Define $M = e\mathbb{C}G$. The following is obvious:

Proposition. The right module M has a basis given by the elements $eg = \frac{1}{|B|}\sum_{b\in Bg} b$ where g ranges over a set of representatives for the distinct right cosets of B in G.

Most of the following is obvious too:

Proposition. The algebra $\mathcal{H}$ has a basis given by the elements $ege = \frac{1}{|BgB|}\sum_{b\in BgB} b$ where g ranges over a set of representatives of the distinct (B, B)-double cosets of G.

Proof. The formula for ege follows by noting that the number of $(b, b') \in B \times B$ with $bgb' = g$ for a given $g \in G$ is $|B|^2/|BgB|$. □

It follow from the homework assignment that $\mathcal{H}$ has no nilpotent ideals. Therefore:

Proposition. $\mathcal{H}$ is semisimple.

This implies as a corollary the following weaker form of Maschke's theorem.

Corollary. $\mathbb{C}G$ is semisimple.

Proof. This is well-known, has more direct proofs, but follows in our case since $\mathbb{C}G = \mathcal{H}(G, \{1\})$. □

The following is the fundamental theorem of (semisimple) Hecke algebras, and explains why considering $\mathcal{H}$ is a worthwhile exercise if one wants to under the simple submodules of M.

Theorem. The map $N \mapsto Ne$ defines a bijection between isomorphism classes of simple G-submodules of M and simple $\mathcal{H}$-modules. The multiplicity of a submodule N in M is the dimension of Ne.

Proof. Since $\mathbb{C}G$ is semisimple, it is isomorphic to an algebra of square block diagonal matrices. Moreover, after changing the basis, we may assume that in this algebra the idempotent e corresponds to a diagonal matrix whose diagonal entries are all 0 or 1: it is a standard linear algebra exercise that any idempotent matrix can be diagonalized and has eigenvalues all 0 or 1. Let I be the set of rows/columns that contain nonzero entries of e. The right module M then consists of the block diagonal matrices $m \in \mathbb{C}G$ with $m_{ij} = 0$ if $i \notin I$. The simple submodules of M are given by $N = v\mathbb{C}G$ where v is a nonzero matrix, with nonzero entries only in a single column, and only in the rows $i \in I$. The elements of such a module N are the block diagonal matrices in $\mathbb{C}G$ in which each column is a scalar multiple of the nonzero column in v. It is evident from this description that Ne is never zero, since N contains elements with nonzero entries in the columns $i \in I$. Therefore, by the results at the end of the last section, Ne is a simple $\mathcal{H}$-module whenever $N \subset M$ is a simple submodule. Conversely, since $\mathcal{H}$ is semisimple, each right simple $\mathcal{H}$-module is isomorphic to $u\mathcal{H}$ for a primitive idempotent $u \in \mathcal{H}$, and $N = u\mathbb{C}G$ is then a simple submodule of M, and we have $Ne = u\mathbb{C}Ge = ue\mathbb{C}Ge = u\mathcal{H}$, so $u\mathcal{H} \mapsto u\mathbb{C}G$ is the inverse to $N \mapsto Ne$.

To justify the assertion about multiplicities of submodules, reconsider the identification of $\mathbb{C}G$ with a matrix algebra. The sum of all submodules of M isomorphic to a given simple submodule $N \subset M$ may be identified with $\mathbb{C}^{m\times d}$, where d is the dimension of N and m is the multiplicity of N in M. The Hecke algebra $\mathcal{H} = e\mathbb{C}Ge$ can be viewed as the direct sum of the matrix algebras $\mathbb{C}^{m_i \times m_i}$ where $m_1, m_2, \dots, m_n$ are the multiplicities of the non-isomorphic simple submodules of M. If $N \subset M$ is a simple submodule with multiplicity m, then eN is a simple submodule of $\mathcal{H}$ occurring within the $m \times m$ block, and so is m-dimensional. □

Bringing things full circle, let $\mathbb{F}_q$ be a finite field of order q and consider the Hecke algebra $\mathcal{H}(G, B)$ where $G = \mathrm{GL}_n(\mathbb{F}_q)$ and $B \subset G$ is the subgroup of invertible upper triangular matrices. Next time, we will show that $\mathcal{H}(G, B)$ is isomorphic to the generic Hecke algebra over $\mathbb{C}$ of the Coxeter group S_n, with parameters $a_s = q - 1$ and $b_s = q$.