1 Three forms Hecke algebras

We've now seen Hecke algebras in three different guises:

1. (Deformation of group algebra of a Coxeter group.) If (W, S) is a Coxeter system, and A is a commutative ring, and $a_s, b_s \in A$ for $s \in S$ are such that $a_s = a_t$ and $b_s = b_t$ when s is conjugate to t in W, then the *generic Hecke algebra* $\mathcal{H}$ is the unique A-algebra structure on the free A-module A-span$\{T_w : w \in W\}$ in which T_1 serves as the unit element and we have $T_w T_s = T_{ws}$ if $\ell(ws) > \ell(w)$ and $T_s^2 = a_s T_w + b_s$ for $s \in S$ and $w \in W$.

   This, equivalently, is the A-algebra generated by T_s for $s \in S$ subject to the relations

   $T_s T_t T_s \cdots = T_t T_s T_t \cdots$ (both sides with $m(s,t)$ factors) for $s, t \in S$ such that $m(s,t) < \infty$.

   $T_s^2 = a_s T_s + b_s$ for $s \in S$.

   Since setting $a_s = 0$ and $b_s = 1$ turns this into a presentation for the group algebra AW, we say that $\mathcal{H}$ is a *deformation* of AW.

   Note in either case that $T_w = T_{s_1} T_{s_2} \cdots T_{s_n}$ if $w = s_1 s_2 \cdots s_n \in W$ is a reduced expression.

2. (Endomorphism algebra of module generated by an idempotent.) If A is a finite-dimensional algebra over a field K and $e = e^2 \in A$ is an idempotent, then the *Hecke algebra* of (A, e) is $\mathcal{H}(A, e) = eAe = \{eae : a \in A\}$. This is an associative A-algebra with unit e.

   Proposition. $\mathcal{H}(A, e) \cong \mathrm{End}_A(eA)$, the algebra of right A-module endomorphisms of eA.

   The algebra A is *semisimple* if no nonzero element annihilates every simple right A-module.

   Theorem (Wedderburn)**.** If A is semisimple then $A \cong \bigoplus_{i=1}^n K^{d_i \times d_i}$ is isomorphic as an algebra to a direct sum of matrix algebras.

3. (Bi-invariant functions on a group.) Let G be a finite group and let $B \subset G$ be a subgroup. The *Hecke algebra* of (G, B) is $\mathcal{H}(G, B) = \mathcal{H}(\mathbb{C}G, e)$ where $e = \frac{1}{|B|}\sum_{b \in B} b$. Elements of this algebra can be identified with functions $f : G \to \mathbb{C}$ with $f(b_1 g b_2) = f(g)$ for all $g \in G$ and $b_1, b_2 \in B$: every element of $\mathcal{H}(G, B)$ has the form $\sum_{g \in G} f(g)g$ for a function with this property.

   Last time we proved:

   Proposition. Both $\mathbb{C}G$ and $\mathcal{H}(G, B)$ are semisimple $\mathbb{C}$-algebras.

   Proposition. $N \mapsto Ne$ defines a bijection between isomorphism classes of simple right G-submodules of $e\mathbb{C}G$ and simple right $\mathcal{H}$-modules. The multiplicity of N in $e\mathbb{C}G$ is the dimension of Ne.

Hecke algebras of type (3) are a special case of those of type (2). Today we will show that a special case of (3) coincides with a special case of (1), thus explaining why the algebra $\mathcal{H}$ attached to (W, S) is called a Hecke algebra.

2 Hecke algebras for (G, B)

Let $\mathbb{F}_q$ be a finite field with q elements. Let n be a positive integer.

Set $G = \mathrm{GL}_n(\mathbb{F}_q)$ and define $B \subset G$ as the subgroup of upper-triangular matrices.

Let $e = \frac{1}{|B|}\sum_{b \in B} b \in \mathbb{C}G$ as usual, and let $\mathcal{H} = \mathcal{H}(G, B) = e\mathbb{C}Ge$.

We identify S_n as a subgroup of G by letting a permutation $w \in S_n$ correspond to the linear transformation with $we_i = e_{w(i)}$ where $e_1, e_2, \ldots, e_n$ are the standard basis elements of $\mathbb{C}^n$.

Proposition. The matrix of $w \in S_n$ under the identification is $\sum_{i=1}^n E_{w(i),i}$ where E_{ij} is the $n \times n$ matrix with 1 in position (i,j) and 0 in all other positions.

Proof. Follows by an easy calculation, noting that $E_{ij}e_k$ is e_i if $j = k$ and otherwise zero. □

Corollary. It holds that $w^{-1} = w^T$ for $w \in S_n \subset G$.

Proof. We compute $w^T w = \left(\sum_{i=1}^n E_{i,w(i)}\right)\left(\sum_{j=1}^n E_{w(j),j}\right) = \sum_{ij} E_{i,w(i)}E_{w(j),j} = \sum_{i=1}^n E_{ii} = 1.$ □

Proposition (Bruhat decomposition in type A)**.** The subgroup $S_n \subset G$ is a complete set of representatives of the double cosets BgB for $g \in G$. I.e., if $g \in G$ then $BgB = BwG$ for a unique $w \in S_n$.

To see how this generalizes, check out `https://en.wikipedia.org/wiki/Bruhat_decomposition`.

Proof. The main idea is to recall Gaussian elimination and use induction. The key thing to understand is what happens to a matrix when we multiply on the left and right by an element of B: rows get rescaled and/or added to rows above, while columns get rescaled and/or added to columns to the right.

For example, if $n = 2$ then the cases for BgB are:

1. $BgB = \begin{pmatrix} * & * \\ * & * \end{pmatrix} = B\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}B.$
2. $BgB = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = B\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}B.$

The details for general n are left as an exercise. □

The set $\{ege : g \in G\}$ clearly forms a basis for $\mathcal{H}$, though it frequently happens that $ege = ehe$ for distinct $g, h \in G$. Since $be = eb = e$ if and only if $b \in B$, it follows that $\{ege : g \in G\} = \{ewe : w \in S_n\}$ and that the elements of the latter set are all distinct. The next few results will tell us how to multiply these basis elements.

Lemma. If $b \in B$ and $w \in S_n$ then $wbw^{-1} \in B$ if and only if $b_{ij} = 0$ for all $(i,j) \in \mathrm{Inv}(w)$.

Proof. Let $b \in B$ and $w \in S_n$. We have

$$(wbw^{-1})_{ji} = e_j^T wbw^{-1}e_i = (w^{-1}e_j)^T b(w^{-1}e_i) = e^T_{w^{-1}(j)}be_{w^{-1}(i)} = b_{w^{-1}(j),w^{-1}(i)}.$$

On the other hand, $wbw^{-1} \in B$ if and only if $(wbw^{-1})_{ji} = 0$ whenever $i < j$.

If $i < j$ and $w^{-1}(i) < w^{-1}(j)$ then the calculation above shows that $(wbw^{-1})_{ji} = b_{w^{-1}(j),w^{-1}(i)} = 0$.

If $i < j$ and $w^{-1}(i) > w^{-1}(j)$ then $(w^{-1}(j), w^{-1}(i)) \in \mathrm{Inv}(w)$, and every inversion of w arises in this way.

We conclude that $(wbw^{-1})_{ji} = 0$ for all $i < j$ if and only if $b_{ji} = 0$ for all $(i,j) \in \mathrm{Inv}(w)$. □

Lemma. Suppose $w \in S_n$ and $s = s_i = (i, i+1)$ are such that $\ell(ws) > \ell(w)$. Then

$$(BwB)(BsB) = \{awb \cdot csd : a, b, c, d \in B\} = BwsB.$$

Proof. Note that $(i, i{+}1) \notin \mathrm{Inv}(w)$ since $\ell(ws) > \ell(w)$. It suffices to check that if $b \in B$ then $wbs \in BwsB$. We confirm this with some slightly imprecise, but hopefully intuitively clear matrix calculations:

$$
\begin{aligned}
wbs &= w \begin{pmatrix} * & * & * & * \\ & a & x & * \\ & 0 & b & * \\ & & & * \end{pmatrix} s \\
&= w \begin{pmatrix} 1 & 0 & 0 & 0 \\ & a & x & 0 \\ & 0 & b & 0 \\ & & & 1 \end{pmatrix} \begin{pmatrix} * & * & * & * \\ & 1 & 0 & * \\ & 0 & 1 & * \\ & & & * \end{pmatrix} s \\
&= \underbrace{w \begin{pmatrix} 1 & 0 & 0 & 0 \\ & a & x & 0 \\ & & b & 0 \\ & & & 1 \end{pmatrix} w^{-1}}_{\in B \text{ by lemma}} ws \underbrace{s \begin{pmatrix} * & * & * & * \\ & 1 & 0 & * \\ & 0 & 1 & * \\ & & & * \end{pmatrix} s}_{\in B \text{ by lemma}} \in BwsB.
\end{aligned}
$$

Here, the rows/columns containing a, b, x are i and $i+1$. □

Lemma. Let $s = s_i \in (i, i+1) \in S_n$. Then $(BsB)(BsB) = B \sqcup BsB$.

Moreover, the number of elements $b \in B$ with $sbs \in B$ is $|B|/q$.

Proof. Let $b \in B$. It suffices to show that $sbs \in B \sqcup BsB$.

By the lemma above, we have $sbs \in B$ if and only if $b_{i,i+1} = 0$. Directly, note that if $b_{i,i+1} = 0$ then

$$
sbs = s \begin{pmatrix} * & * & * & * \\ & a & 0 & * \\ & 0 & b & * \\ & & & * \end{pmatrix} s = \begin{pmatrix} * & * & * & * \\ & b & 0 & * \\ & 0 & a & * \\ & & & * \end{pmatrix} \in B.
$$

On the other hand if $b_{i,i+1} \neq 0$ then

$$
sbs = s \begin{pmatrix} * & * & * & * \\ & a & x & * \\ & 0 & b & * \\ & & & * \end{pmatrix} s = \begin{pmatrix} * & * & * & * \\ & b & 0 & * \\ & x & a & * \\ & & & * \end{pmatrix} \in B \begin{pmatrix} * & * & * & * \\ & 0 & 1 & * \\ & 1 & 0 & * \\ & & & * \end{pmatrix} B = BsB.
$$

Thus $sbs \in B$ if and only if $b_{i,i+1} = 0$, and otherwise $sbs \in BsB$, so the lemma follows. □

Define $T_w = q^{\ell(w)} ewe$ for $w \in S_n \subset G$. These elements are a basis for $\mathcal{H}$.

Theorem. If $w \in S_n$ and $s \in S = \{s_1, s_2, \dots, s_{n-1}\}$ then

(a) $T_w T_s = T_{ws}$ if $\ell(ws) > \ell(w)$.

(b) $T_s^2 = (q-1)T_s + qT_1$.

Thus $\mathcal{H} = \mathcal{H}(G, B)$ is the generic Hecke algebra of the Coxeter system $(W, S) = (S_n, \{s_1, s_2, \dots, s_{n-1}\})$ with $A = \mathbb{C}$, $a_s = q - 1$, and $b_s = q$.

Proof. If $\ell(ws) > \ell(w)$ then, using the above lemmas, we compute

$$
T_w T_s = q^{\ell(w)} ewe \cdot qese = q^{\ell(w)+1} \frac{1}{|B|} \sum_{b \in B} ewbse = q^{\ell(ws)} ewse = T_{ws}
$$

as desired. Likewise, we have

$$T_s^2 = q^2 \frac{1}{|B|} \sum_{b\in B} esbse = \frac{q^2}{|B|}\left(\tfrac{|B|}{q} e + \left(|B| - \tfrac{|B|}{q}\right) ese\right) = qe + (q^2-q)ese = (q-1)T_s + qT_1.$$

□

The results in this section generalize significantly. The generic Hecke algebra of any finite Weyl group with $A = \mathbb{C}$, $a_s = q-1$, and $b_s = q$ for a prime power q may be realized as a Hecke algebra $\mathcal{H}(G,B)$ where G is a finite group with a so-called *BN-pair.*

These results motivate us to specifically consider generic Hecke algebras with parameters $a_s = q-1$ and $b_s = q$. We start this in the next section, and continue next time.

3 Iwahori-Hecke algebras

Let (W,S) be a Coxeter system.

Let $A = \mathbb{Z}[x,x^{-1}]$ be the ring of Laurent polynomials in one variable x.

Set $a_s = x^2-1$ and $b_s = x^2$ for $s \in S$.

The *Iwahori-Hecke algebra* of (W,S) is the generic Hecke algebra $\mathcal{H}$ defined with respect to these choices of A and parameters a_s and b_s. This algebra is the free A-module with basis $\{T_w : w \in W\}$, with

$$T_1 = 1 \qquad \text{and} \qquad T_wT_s = T_{ws} \text{ if } ws > w \qquad \text{and} \qquad T_s^2 = (x^2-1)T_s + x^2T_1$$

for $s \in S$ and $w \in W$.

We imagine that $x^2 = q$ in the previous section. For technical reasons which are hard to motivate right now, we need A is contain square root of q; hence our choice of parameters. Many subsequent formulas become nicer if we work not with the basis elements $T_w \in \mathcal{H}$, but rather the rescaled elements

$$H_w = x^{-\ell(w)}T_w \qquad \text{for } w \in W.$$

Since x is invertible in $\mathcal{H}$, the elements $\{H_w : w \in W\}$ are also an A-basis for the algebra. We observe some key properties of this new basis:

Proposition. Let $s \in S$ and $w \in W$.

(1) $H_1 = T_1 = 1 \in \mathcal{H}$.

(2) $H_s^2 = 1 + (x-x^{-1})H_s$ and so $(H_s - x)(H_s + x^{-1}) = 0$.

(3) $H_sH_w = \begin{cases} H_{sw} & \text{if } sw > w \\ H_{sw} + (x-x^{-1})H_w & \text{if } sw < w \end{cases}$ and $H_wH_s = \begin{cases} H_{ws} & \text{if } ws > w \\ H_{ws} + (x-x^{-1})H_w & \text{if } sw < w. \end{cases}$

Proof. Part (1) is trivial, part (2) holds since

$$H_s^2 = x^{-2}T_s^2 = x^{-2}(x^2-1)T_s + x^{-2}x^2T_1 = (x-x^{-1})xT_s + T_1 = (x-x)^{-1}H_s + 1,$$

and part (3) follows by similar calculations: for example, if $sw > w$ then

$$H_sH_w = x^{-\ell(w)-1}T_sT_w = x^{-\ell(sw)}T_{sw} = H_{sw}.$$

□

Of course, we could achieve the same effect by using the usual basis elements T_w with different parameters (namely, $a_s = x - x^{-1}$ and $b_s = 1$) in place of H_w. This would conflict with the common usage of T_w in the literature and in Humphreys's book, however.

Corollary. Each $H_s \in \mathcal{H}$ for $s \in S$ is invertible, with inverse $H_s^{-1} = H_s + x^{-1} - x$.

Proof. Note that $H_s(H_s + x^{-1} - x) = H_s^2 - (x - x^{-1})H_s = H_1$. □

Next time: the bar involution of $\mathcal{H}$ and its Kazhdan-Lusztig basis.