1 Last time: forms of Hecke algebras

Let (W, S) be a Coxeter system, set $A = \mathbb{Z}[x, x^{-1}]$, and recall that the *Iwahori-Hecke algebra* $\mathcal{H}$ of (W, S) is the unique A-algebra structure on the free A-module with basis $\{H_w : w \in W\}$ in which, for $w \in W$ and $s \in S$, it holds that

(1) $H_1 = 1$.

(2) $H_s^2 = H_1 + (x - x^{-1})H_s$.

(3) $H_s H_w = H_{sw}$ if $sw > w$ and $H_w H_s = H_{ws}$ if $ws > w$.

Last time we saw that this algebra arises, when $x = \sqrt{q}$ for a prime power q and $W = S_n$, as the more general *Hecke algebra* $\mathcal{H}(G, B) = e\mathbb{C}Ge$ where $G = \mathrm{GL}_n(\mathbb{F}_q)$, $B \subset G$ is the subgroup of upper triangular matrices, and $e = \frac{1}{|B|}\sum_{b \in B} b \in \mathbb{C}G$.

A corollary of the results last time shows that the irreducible constituents of the induced representation $\mathrm{Ind}_B^G(\mathbf{1}) \cong e\mathbb{C}G$ are in bijection with the irreducible submodules of $\mathcal{H}(G, B)$, with multiplicities in the first case corresponding to dimensions in the second. This is one reason to be interested in $\mathcal{H}(G, B)$.

2 The bar involution

Continuing from last time:

Proposition. Each $H_s \in \mathcal{H}$ for $s \in S$ is invertible, with inverse $H_s^{-1} = H_s + x^{-1} - x$.

Proof. Note that $H_s(H_s + x^{-1} - x) = H_s^2 - (x - x^{-1})H_s = H_1$. □

Corollary. H_w is invertible for all $w \in W$.

Proof. This holds since $H_w = H_{s_1} H_{s_2} \cdots H_{s_n}$ if $w = s_1 s_2 \cdots s_n \in W$ is any reduced expression. □

Remark. Note that the solutions to $\zeta^2 = (x - x^{-1})\zeta + 1$ are $\zeta = x$ and $\zeta = -x^{-1}$. Hence $H_w \mapsto x^{\ell(w)}$ and $H_w \mapsto (-x)^{-\ell(w)}$ both define A-algebra homomorphisms $\mathcal{H} \to A = \mathbb{Z}[x, x^{-1}]$.

Remark. Since $H_s = x^{-1}T_s$ in our earlier notation, $\mathcal{H}$ is the A-algebra generated by H_s $(s \in S)$ subject to the relations

1. $(H_s - x)(H_s + x^{-1}) = 0$ for $s \in S$.
2. $H_s H_t H_s \cdots = H_t H_s H_t \cdots$, both sides with $m(s, t)$ factors, for $s, t \in S$.

Proposition. There is a unique A-algebra automorphism of $\mathcal{H}$ with $H_s \mapsto -H_s^{-1}$ for $s \in S$.

Proof. Just check that the proposed map preserves the relations in the previous remark: this implies that the map has a unique extension to an invertible A-algebra homomorphism $\mathcal{H} \to \mathcal{H}$.

For example, we still have $(-H_s^{-1} - x)(-H_s^{-1} + x^{-1}) = (-H_s + x^{-1})(-H_s + x) = (H_s - x)(H_s + x^{-1}) = 0$ for $s \in S$. Likewise, $(-H_s^{-1})(-H_t^{-1})(-H_s^{-1}) \cdots = (-H_t^{-1})(-H_s^{-1})(-H_t^{-1}) \cdots$ holds since inverting both sides and canceling signs gives the original relation $H_s H_t H_s \cdots = H_t H_s H_t \cdots$. □

Proposition. There exists a unique ring automorphism of $\mathcal{H}$ with $x \mapsto x^{-1}$ and $H_s \mapsto -H_s$ for $s \in S$.

Note that an A-algebra automorphism is required to be A-linear, but a ring automorphism is only required to be $\mathbb{Z}$-linear.

Proof. The ring automorphism $\alpha : \mathcal{H} \to \mathcal{H}$ with these properties must satisfy $\alpha\left(\sum_{w\in W} h_w H_w\right) = \sum_{w\in W} h_w(x^{-1})(-1)^{\ell(w)} H_w$ for $h_w \in A$. To show that this map is a ring homomorphism, it suffices to check that $\alpha(H_s H_w) = \alpha(H_s)\alpha(H_w) = -(-1)^{\ell(w)} H_s H_w$ for $s \in S$ and $w \in W$. This is straightforward from the defining properties (1)-(3) of $\mathcal{H}$ in Section 1. For example, if $sw < w$ then

$$\alpha(H_s H_w) = \alpha(H_{sw} + (x - x^{-1})H_w) = -(-1)^{\ell(w)} H_{sw} + (x^{-1} - x)(-1)^{\ell(w)} H_w = -(-1)^{\ell(w)} H_s H_w.$$

□

Composing these maps gives the involution of $\mathcal{H}$ that we will be most interested in:

Corollary. There exists a unique ring involution $\mathcal{H} \to \mathcal{H}$ with $H_s \mapsto H_s^{-1}$ for $s \in S$ and $x \mapsto x^{-1}$.

Denote this ring involution by $h \mapsto \overline{h}$ for $h \in \mathcal{H}$.

Call this the *bar involution/operator* of $\mathcal{H}$.

Proof. Composing the previous two maps gives a ring automorphism with

$$H_s \mapsto -H_s^{-1} = -H_s + x - x^{-1} \mapsto H_s + x^{-1} - x = H_s^{-1}$$

and $x \mapsto x \mapsto x^{-1}$, as desired. □

Note that $\overline{H_w} = H_{w^{-1}}^{-1}$ since if $H_w = H_{s_1} \cdots H_{s_k}$ $(s_i \in S)$ then

$$\overline{H_w} = \overline{H_{s_1}} \cdots \overline{H_{s_k}} = H_{s_1}^{-1} \cdots H_{s_k}^{-1} = (H_{s_k} \cdots H_{s_1})^{-1}.$$

Thus $\overline{\sum_{w\in W} h_w H_w} = \sum_{w\in W} h_w(x^{-1}) H_{w^{-1}}^{-1}$ for any coefficients $h_w \in A = \mathbb{Z}[x, x^{-1}]$.

The main property of the bar involution is the following:

Proposition. If $w \in W$ then $\overline{H_w} \in H_w + \sum_{v<w} AH_v$, where $<$ is the Bruhat order on W.

Proof. Since $\overline{H_1} = H_1$, the claim is trivial if $\ell(w) = 0$.

Suppose $\ell(w) > 0$ and choose $s \in S$ with $sw < w$. By induction, we may assume that

$$\overline{H_w} = \overline{H_s} \cdot \overline{H_{sw}} \in (H_s + x^{-1} - x)\left(H_{sw} + \sum_{v<sw} AH_v\right) \subset H_w + \sum_{v<sw} AH_s H_v + \sum_{v<w} AH_v.$$

Note that if $v < sw$ and $sv > v$ then $sv < w$: this follows either by the lifting property or the subexpression characterization of the Bruhat order. If $v < sw$ and $sv < v$ then again $sv < v < sw < w$. Thus we have $H_s H_v \in \sum_{u<w} AH_u$ whenever $v < sw$, so the result follows. □

Lemma. If $h = \overline{h} \in x^{-1}\mathbb{Z}[x^{-1}]\text{-span}\{H_w : w \in W\}$ then $h = 0$.

Proof. Suppose h is a nonzero element of $x^{-1}\mathbb{Z}[x^{-1}]\text{-span}\{H_w : w \in W\}$. Write $h = \sum_{w\in W} h_w H_w$ where $h_w \in x^{-1}\mathbb{Z}[x^{-1}]$. Choose w which is maximal in Bruhat order from the set $\{w \in W : h_w \neq 0\}$. In view of the maximality of this choice, it follows from the previous proposition that $\overline{h_w} = h_w(x^{-1}) \neq h_w$ is the coefficient of H_w in $\overline{h}$, so $h \neq \overline{h}$. □

3 Kazhdan-Lusztig basis

Having introduced the bar involution of $\mathcal{H}$, we can now characterize an important second basis of $\mathcal{H}$.

Theorem (Kazhdan and Lusztig (1979))**.** For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ with

$$\overline{C_w} = C_w \in H_w + \sum_{v<w} x^{-1}\mathbb{Z}[x^{-1}]H_v.$$

The set $\{C_w : w \in W\}$ is an A-basis for $\mathcal{H}$, called the *Kazhdan-Lusztig (KL) basis* or *canonical basis*.

Proof. The uniqueness property is immediate from our last lemma: if C'_w were another element with the same properties then $C_w - C'_w$ we be an element of $x^{-1}\mathbb{Z}[x^{-1}]$-span$\{H_w : w \in W\}$ invariant under the bar involution, so $C_w - C'_w = 0$ and $C_w = C'_w$.

To prove the existence of the basis element C_w, first let $C_1 = H_1 = 1$ and $C_s = H_s + x^{-1}$ for $s \in S$. Note that $\overline{C_1} = C_1$ and $\overline{C_s} = H_s + x^{-1} - x + x = C_s$. Also observe that

$$C_s H_w = \begin{cases} H_{sw} + x^{-1}H_w & \text{if } sw > w \\ H_{sw} + xH_w & \text{if } sw < w \end{cases} \qquad (*)$$

for $s \in S$ and $w \in W$.

Fix $w \in W$ with $\ell(w) \geq 2$. Assume C_v is given for $v < w$.

Then $C_s C_{sw} \in H_w + x^{-1}H_{sw} + \sum_{v<sw} x^{-1}\mathbb{Z}[x^{-1}]C_s H_v$.

Note, as earlier, that if $v < sw$ then $v < w$ and $sv < w$.

Therefore (*) implies that $C_s C_{sw} = H_w + \sum_{v<w} \overline{h_v} H_v$ for some polynomials $h_v \in \mathbb{Z}[x]$.

Define $C_w = C_s C_{sw} - \sum_{v<w} h_v(0) C_v$.

By construction, C_w has the desired properties. The uniqueness proved earlier implies that this construction of C_w is well-defined, independent of the choices of s. □

Write $C_w = \sum_{y \in W} h_{yw} Hy$ where $h_{y,w} \in \mathbb{Z}[x^{-1}]$ and define $P_{yw} = x^{\ell(w)-\ell(y)} h_{yw}$ for $y, w \in W$.

Some notable properties (some easy, some less so) that will be shown next time:

Proposition. Let $y, w \in W$.

1. $P_{yw} \in \mathbb{Z}[x^2]$.
2. P_{yw} has constant term 1 if $y \leq w$, and $P_{ww} = 1$.
3. If $y \not\leq w$ then $P_{yw} = 0$.
4. P_{yw} has (even) degree at most $\ell(w) - \ell(y) - 1$.
5. $P_{yw} = P_{y^{-1}w^{-1}}$.

The polynomials P_{yw} are called the *Kazhdan-Lusztig polynomials* of (W, S).

Why is the KL basis interesting? The original motivation came from representation theory, specifically the *Kazhdan-Lusztig conjectures* (1979), which are paraphrased informally as follows.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The *Verma modules* of $\mathfrak{g}$ are certain highest weight modules M_λ. Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ be half the sum of the positive roots in the root system Φ of the Weyl group W of $\mathfrak{g}$ (which is a finite reflection group). Let $M_w = M_{-w\rho-\rho}$ for each $w \in W$. Let L_w be the irreducible quotient of M_w: this is the simple highest weight $\mathfrak{g}$-module of highest weight $-w\rho - \rho$. Finally write $\mathrm{ch}(M)$ for the character of a highest weight $\mathfrak{g}$-module M.

The modules M_w and L_w are easy to construct, and it was expected that it would also be easy to express how one set of modules decomposes as (formal) linear combination of the other set. This problem turns out to be quite nontrivial from an algebraic standpoint, however. Kazhdan and Lusztig proposed the first computable method of obtaining this decomposition. Their conjecture is notable for its simple solution to this open problem (the definition of the KL polynomials requires little advanced theory beyond the definition of the Bruhat order on a Coxeter group and some analysis of the definition of $\mathcal{H}$), and the difficulty of its proof.

Conjecture (Kazhdan and Lusztig (1979))**.** The decomposition of the Verma modules M_w into simple modules L_w and vice versa is precisely determined by the values of the Kazhdan-Lusztig polynomials at $x = 1$. Specifically, for $w \in W$ it holds that

$$\mathrm{ch}(L_w) = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} P_{yw}(1)\mathrm{ch}(M_y) \qquad \text{and} \qquad \mathrm{ch}(M_w) = \sum_{y \leq w} P_{w_0 w, w_0 y}(1)\mathrm{ch}(L_y)$$

where w_0 is the longest element of the finite group W.

The KL conjectures were proved independently by Beilinson and Bernstein, and Brylinski and Kashiwara in 1981. Their similar proofs brought many ideas from algebraic geometry to the fore of representation theory, and stimulated the development of geometric representation theory over the next few decades.