1 Last time: the Kazhdan-Lusztig basis of $\mathcal{H}$

Let (W, S) be a Coxeter system

Let $\mathcal{H}$ be the Iwahori-Hecke algebra of (W, S).

Recall that this is the free $\mathbb{Z}[x, x^{-1}]$-module with basis H_w for $w \in W$ and the unique $\mathbb{Z}[x, x^{-1}]$-algebra structure in which $H_1 = 1$, $H_u H_v = H_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$, and $H_s^2 = 1 + (x - x^{-1})H_s$ for $s \in S$.

Recall that $H_s^{-1} = H_s + x^{-1} - x$ for $s \in S$. Last time we proved:

Proposition. There exists a unique ring automorphism $h \mapsto \overline{h}$ of $\mathcal{H}$ with

$$\overline{x} = x^{-1} \qquad \text{and} \qquad \overline{H_s} = H_s^{-1} \text{ for } s \in S.$$

We call this the *bar involution* of $\mathcal{H}$.

Since the bar involution is a ring automorphism, it holds that $\overline{gh} = \overline{g} \cdot \overline{h}$ and $\overline{g+h} + \overline{g} + \overline{h}$ for all $g, h \in \mathcal{H}$. It follows that $\overline{\overline{h}} = h$ for $h \in \mathcal{H}$, and $\overline{H_w} = (H_{w^{-1}})^{-1}$ for $w \in W$.

Write $<$ for the Bruhat order on W. There two main results last time:

Proposition. If $w \in W$ then $\overline{H_w} \in H_w + \sum_{v<w} \mathbb{Z}[x, x^{-1}]H_v$, where $<$ is the Bruhat order on W.

Theorem (Kazhdan and Lusztig (1979))**.** For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ with

$$\overline{C_w} = C_w \in H_w + \sum_{v<w} x^{-1}\mathbb{Z}[x^{-1}]H_v.$$

The set $\{C_w : w \in W\}$ is a $\mathbb{Z}[x, x^{-1}]$-basis for $\mathcal{H}$, called the *Kazhdan-Lusztig (KL) basis* or *canonical basis*.

2 Kazhdan-Lusztig polynomials

Today, we discuss how to actually compute the KL basis and its structure constants. It is not difficult to see that $C_1 = H_1 = 1$ and $C_s = H_s + x^{-1}$ for $s \in S$. Therefore

$$C_s H_w = \begin{cases} H_{sw} + x^{-1}H_w & \text{if } sw > w \\ H_{sw} + xH_w & \text{if } sw < w \end{cases}$$

for $s \in S$ and $w \in W$.

Define $h_{yw} \in \mathbb{Z}[x^{-1}]$ for $y, w \in W$ such that $C_w = \sum_{y \in W} h_{yw} H_y$.

Define $\mu(y, w)$ as the coefficient of x^{-1} in h_{yw}. Note that $\mu(y, w) \neq 0$ only if $y < w$.

Theorem. Let $w \in W$ and $s \in S$. Then

$$C_s C_w = \begin{cases} (x + x^{-1})C_w & \text{if } sw < w \\ \\ C_{sw} + \displaystyle\sum_{\substack{y \in W \\ sy<y<w}} \mu(y, w)C_y & \text{if } sw > w. \end{cases}$$

In particular, $C_{sw} = C_s C_w - \sum_{sy<y<w} \mu(y, w)C_y$ if $sw > w$.

Proof. Assume $sw > w$. You can check that the element defined by

$$C'_{sw} = C_s C_w - \sum_{sy<y<w} \mu(y,w) C_y \in \mathcal{H}$$

satisfies $\overline{C'_{sw}} = C'_{sw} \in H_{sw} + \sum_{y<sw} x^{-1}\mathbb{Z}[x^{-1}]H_y$, so by uniqueness of the KL basis, $C_{sw} = C'_{sw}$. Alternatively suppose $sw < w$. If $w = s$ then we have

$$C_s C_w = C_s^2 = H_s^2 + 2x^{-1}H_s + x^{-2} = (x - x^{-1} + 2x^{-1})H_s + x^{-2} + 1 = (x + x^{-1})C_s.$$

Assume $C_s C_v = (x + x^{-1})C_v$ if $sv < v < w$. Then, using the first part of the proof, we have

$$\begin{aligned} C_s C_w &= C_s \left(C_s C_{sw} - \sum_{sy<y<sw} \mu(y,sw) C_y \right) \\ &= C_s^2 C_{sw} - \sum_{sy<y<sw} \mu(y,sw) C_s C_y \\ &= (x + x^{-1}) \left(C_s C_{sw} - \sum_{sy<y<sw} \mu(y,sw) C_y \right) = (x + x^{-1}) C_w. \end{aligned}$$

□

Corollary. Let $s \in S$ and $y, w \in W$ with $sw > w$. Set $c = \ell(y) - \ell(sy)$. Then

$$h_{y,sw} = x^c h_{yw} + h_{sy,w} - \sum_{\substack{z\in W \\ y\leq z<w \\ sz<z}} \mu(z,w) h_{yz}.$$

Proof. The identity $C_{sw} = C_s C_w - \sum_{sz<z<w} \mu(z,w) C_z$ implies that

$$\sum_{y\in W} h_{y,sw} H_y = \sum_{y\in W} h_{yw} C_s H_y - \sum_{sz<z<w} \sum_{y\leq z} \mu(z,w) h_{yz} H_y \tag{*}$$

and we have

$$\begin{aligned} \sum_{y\in W} h_{yw} C_s H_y &= \sum_{sy<y\in W} (x h_{yw} H_y + h_{yw} H_{sy}) + \sum_{sy>y\in W} (x^{-1} h_{yw} H_y + h_{yw} H_{sy}) \\ &= \sum_{y\in W} (x^c h_{yw} + h_{sy,w}) H_y. \end{aligned}$$

The result follows by comparing coefficients of H_y on both sides of (*). □

Recall that $P_{yw} = x^{\ell(w)-\ell(y)} h_{yw} \in \mathbb{Z}[x, x^{-1}]$ for $y, w \in W$.

Corollary. Let $s \in S$ and $y, w \in W$ with $sw > w$. Set $c = \ell(y) - \ell(sy)$. Then

$$\boxed{P_{y,sw} = x^{1+c} P_{yw} + x^{1-c} P_{sy,w} - \sum_{\substack{z\in W \\ y\leq z<w \\ sz<z}} \mu(z,w) x^{\ell(w)-\ell(z)+1} P_{yz}}$$

Proof. Multiply both sides of the previous corollary by $x^{\ell(sw)-\ell(y)} = x^{\ell(w)-\ell(y)+1}$. □

Example. Suppose W is a dihedral group, so that $S = \{a, b\}$ has two elements. In this case, we have $y < w$ if and only if $\ell(y) < \ell(w)$. Given this fact, one can show by induction (using the boxed formula above) that $P_{yw} = 1$ if $y \leq w$ and otherwise $P_{yw} = 0$.

The following properties hold by the definition of C_w.

Fact. $P_{ww} = 1$ and $P_{yw} = 0$ if $y \not\leq w$.

Proposition. $P_{yw} \in \mathbb{Z}[x^2]$ for all $y, w \in W$.

Proof. If $P_{zw} \in \mathbb{Z}[x^2]$ then $\mu(z,w) = 0$ whenever $\ell(w) - \ell(z)$ is even, since otherwise P_{zw} would have have odd degree $\ell(w) - \ell(z) - 1$. Given this, the result follows by induction from the boxed formula. ☐

Corollary. If $y < w$ then the degree of P_{yw} is even and at most $\ell(w) - \ell(y) - 1$.

Proof. This holds since $h_{yw} \in x^{-1}\mathbb{Z}[x^{-1}]$ if $y < w$. ☐

Proposition. P_{yw} has constant term 1 if $y \leq w$.

Proof. This follows by induction on setting $x = 0$ in the boxed formula. ☐

This following fact may be useful on the homework assignment:

Corollary. Let $y, w \in W$ and $s \in S$. If $y < w$, $sw < w$, and $y < sy$ then $P_{yw} = P_{sy,w}$.

Proof. Compare coefficients of H_y on either side of the identity $C_sC_w = (x + x^{-1})C_w$. ☐

We saw last time that the values of the *Kazhdan-Lusztig (KL) polynomials* P_{yw} at $x = 1$ give the multiplicities of the characters of simple highest weight modules in Verma modules and vice versa.

These polynomials are also noteworthy for satisfying much stronger properties which, unlike the ones we've shown so far, do not seem to have any simple algebraic proof.

Theorem (Elias and Williamson (2013))**.** Each $P_{yw} \in \mathbb{N}[x^2]$ has nonnegative coefficients.

This was shown earlier in the case when W is a Weyl group, by identifying the coefficients of P_{yw} with the (necessarily positive) dimensions of certain intersection cohomology groups attached to an associated reductive group. The result for the much larger class of arbitrary Coxeter groups is harder.

Theorem (Elias and Williamson (2013))**.** If $y, z \in W$ then $C_yC_z \in \mathbb{N}[x, x^{-1}]\text{span}\{C_w : w \in W\}$.

I.e., the structure constants for multiplication in the KL basis also have nonnegative coefficients.

Note that this theorem is a consequence of the previous theorem when $y \in S$, by the formula for C_sC_w.

The proofs of these theorems involve identifying $\mathcal{H}$ with the split Grothendieck group of a certain abelian category of graded (bi)modules, such that multiplication in $\mathcal{H}$ corresponds to tensor products of modules, and KL basis elements correspond to indecomposable objects. The positivity of the structure constants decomposing C_yC_z then follows (roughly) from the fact that the tensor product of any two modules is isomorphic to a direct sum of indecomposable objects by definition. A full investigation of this approach is beyond the scope of this course, but we may sketch some of the details in later lectures.

There are some combinatorial formulas for P_{yw} in special cases but it seems unlikely that any general formula can exist. KL polynomials can be arbitrarily complex, even in type A:

Theorem (Polo)**.** Any polynomial in x^2 with positive integer coefficients and constant term 1 occurs as a KL polynomial P_{yw} for y, w in some symmetric group S_n.

We mention a still open conjecture related to the KL polynomials.

Given $y, w \in W$, write $[y, w]$ for the poset $\{v \in W : y \leq v \leq w\}$, ordered by $<$.

Conjecture (Combinatorial invariance)**.** If (W,S) and (W',S') are Coxeter systems and $y,w \in W$ and $y',w' \in W'$ then $P_{yw} = P_{y'w'}$ whenever the intervals $[y,w]$ and $[y',w']$ are isomorphic posets.

This is known to hold at least when $\ell(w) - \ell(y) \leq 4$, if $[y,w]$ is a lattice, or if $y = y' = 1$. There is disagreement about whether this conjecture *should* be true. A proof would be more surprising than a counterexample, though that would be noteworthy too.

3 Left, right, and two-sided cells

Let $y,w \in W$. Define $L(w) = \{s \in S : sw < w\}$ and $R(w) = \{s \in S : ws < w\}$.

Write $y \sim w$ if $\mu(y,w) \neq 0$ or $\mu(w,y) \neq 0$.

Write $y \leq_L w$ if there exists a chain $y = y_0 \sim y_1 \sim \cdots \sim y_r = w$ such that $L(y_i) \not\subset L(y_{i+1})$ for each i.

Write $y \sim_L w$ if $y \leq_L w$ and $w \leq_L y$.

Then $\sim_L$ is an equivalence relation on W, and its equivalence classes are the *left cells* of W.

Similarly, write $y \leq_R w$ if there exists $y = y_0 \sim y_1 \sim \cdots \sim y_r = w$ such that $R(y_i) \not\subset R(y_{i+1})$ for each i.

Write $y \sim_R w$ if $y \leq_R w$ and $w \leq_R y$.

The equivalence classes in W under $\sim_R$ are the *right cells* of W.

Finally, write $y \leq_{LR} w$ if there are $y = y_0, y_1, \dots, y_r = w$ with $y_i \leq_L y_{i+1}$ or $y_i \leq_R y_{i+1}$ for each i.

Write $y \sim_{LR} w$ if $y \leq_{LR} w$ and $w \leq_{LR} y$.

The equivalence classes in W under $\sim_{LR}$ are the *two-sided cells* in W.

These definitions are a little technical, but note that all you need to compute these relations are the values of the KL polynomials P_{yw}. The homework will give you some instructive practice with these types of calculations.

The formula we've given for the product $C_s C_w$ shows that the left cells in W correspond to certain left ideals in $\mathcal{H}$. A similar property holds for the right/two-sided cells.

In detail, let $\mathscr{C}$ be a left cell in W.

Define $\mathcal{I} = \mathbb{Z}[x,x^{-1}]$-span$\{C_w : w \leq_L w' \text{ for some } w' \in \mathscr{C}\}$ and $\mathcal{J} = \mathbb{Z}[x,x^{-1}]$-span$\{C_w \in \mathcal{I} : w \notin \mathscr{C}\}$.

Proposition. Both $\mathcal{I}$ and $\mathcal{J}$ are left ideals in $\mathcal{H}$.

The quotient $\mathcal{I}/\mathcal{J}$ is the *left cell representation* of $\mathscr{C}$. This left $\mathcal{H}$-module is free as a $\mathbb{Z}[x,x^{-1}]$, with a basis given by the images of C_w for $w \in \mathscr{C}$. We define right and two-sided cell representations similarly.

More about cells and other topics next time.