1 Review from the beginning

Given the long break since our last lecture, now seems like a good time to give a quick overview of the main things we've covered in the whole course so far.

1.1 Finite reflection groups

A *reflection* in a finite-dimensional real vector space V with a positive definite bilinear form $(\cdot,\cdot)$ is a linear transformation of the form $s_\alpha : v \mapsto v - 2\frac{(\alpha,v)}{(\alpha,\alpha)}\alpha$ where $\alpha \in V \setminus 0$. Note that $s_\alpha \in O(V) \subset \mathrm{GL}(V)$.

A *finite reflection group* is a finite group generated by some set of reflections $s_\alpha \in O(V)$.

Let W be a finite reflection group. Then $\Phi = \{\alpha \in V \setminus 0 : (\alpha,\alpha) = 1 \text{ and } s_\alpha \in W\}$ is a *root system* preserved by W. Conversely, if Φ is a root system in V then the set of reflections $\{s_\alpha : \alpha \in \Phi\}$ generates a finite reflection group.

Let Φ be a root system with associated reflection group W. If $<$ is a total order on V and $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$ is the corresponding *positive system*, then Φ^+ contains a unique *simple system* Π, and it holds that $W = \langle S\rangle$ where $S = \{s_\alpha : \alpha \in \Pi\}$. Moreover, in this case (W,S) is a *Coxeter system*, meaning that

1. Each $s \in S$ has order two, so $s^2 = 1$.
2. $W = \langle s \in S : (st)^{m(s,t)} = 1 \text{ for } s,t \in S \text{ with } m(s,t) < \infty\rangle$, where $m(s,t)$ is the order of $st \in W$.

The *Coxeter graph* or *diagram* of (W,S) is the weighted, undirected graph on the vertex set S with an edge from s to t labeled by m whenever $s,t \in S$ satisfy $m = m(s,t) > 2$.

The group W is *irreducible* if its graph is connected.

The are four countably infinite "classical" families of irreducible finite reflection groups:

1. Type A_n: if W has Coxeter diagram ∘—∘— ⋯ —∘ where all unlabeled edges have weight 3 then $W \cong S_{n+1}$.
2. Type B_n: if W has Coxeter diagram ∘— ⋯ —∘═∘ where the last edge has weight 4 then W is isomorphic to the centralizer of the reverse permutation in S_{2n}.
3. Type D_n: if W has Coxeter diagram ∘— ⋯ —∘—∘ (with an extra vertex ∘ attached above the second-to-last vertex) then W is isomorphic to a subgroup of index two in the group of type B_n.
4. Type $I_2(m)$: if W has Coxeter diagram ∘$\overset{m}{═}$∘ then W is a dihedral group of order $2m$.

There are six "exceptional" irreducible finite reflection groups which remain: these are referred to as the Coxeter groups of type E_6, E_7, E_8, F_4, H_3, and H_4.

1.2 Coxeter groups

The fact that all finite refection groups are Coxeter groups motivates us to study Coxeter systems abstractly, rather than always working with groups acting on a fixed vector space.

Let (W,S) be a Coxeter system. Assume S is finite.

Define V as the real vector space with a basis given by the symbols α_s for $s \in S$.

Define $(\alpha_s,\alpha_t) = -\cos(\pi/m(s,t))$ for $s,t \in S$ and extend $(\cdot,\cdot)$ to a bilinear form $V \times V \to \mathbb{R}$.

Define $\rho : S \to \mathrm{GL}(V)$ by $\rho(s)v = v - 2(\alpha_s,v)\alpha_s$ for $v \in V$. The map ρ has a unique extension to an injective homomorphism $W \to \mathrm{GL}(V)$ which preserves $(\cdot,\cdot)$. Call this the *geometric representation* of W.

Write wv instead of $\rho(w)v$ for $w \in W$ and $v \in V$, in order to view V as a W-module.

Let $\Phi = \{w\alpha_s : w \in W \text{ and } s \in S\}$. This is the *root system* of (W, S). Unlike the root system of a finite reflection group, this set may be infinite. Define Φ^+ (Φ^-) as the subsets of Φ consisting of the linear combinations of α_s for $s \in S$ with all nonnegative (nonpositive) coefficients. Then $\Phi = \Phi^+ \sqcup \Phi^-$ and for any $w \in W$, the number of $\alpha \in \Phi^+$ with $w\alpha \in \Phi^-$ is the same as the minimum number of factors $s_i \in S$ needed to express $w = s_1 s_2 \cdots s_k$. We denote this *length* by $\ell(w)$.

Let $T = \{wsw^{-1} : s \in S,\ w \in W\}$.

Strong exchange condition. if $w = s_1 s_2 \cdots s_k$ ($s_i \in S$) and $t \in T$ and $\ell(wt) < \ell(w)$ then $wt = s_1 \cdots \widehat{s_i} \cdots s_k$ for some index i, which is unique if $\ell(w) = k$. It follows that if $w = s_1 \cdots s_k$ ($s_i \in S$) and $\ell(w) < k$ then $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k$ for some $1 \leq i < j \leq k$.

If $J \subset S$ and $W_J = \langle J \rangle \subset W$ then (W_J, J) is a Coxeter system with length function $\ell|_{W_J}$.

Matsumoto's theorem. The set of reduced expressions $w = s_1 s_2 \cdots s_k$ ($s_i \in S$) for $w \in W$ is spanned and preserved by the *braid transformations*

$$s_1 \cdots s_i \underbrace{ststststst\cdots}_{m(s,t)\text{ factors}} s_j \cdots s_k \ \leftrightarrow\ s_1 \cdots s_i \underbrace{tststststs\cdots}_{m(s,t)\text{ factors}} s_j \cdots s_k.$$

Classification of finite Coxeter groups. Let (W, S) be a Coxeter group. The following are equivalent:

1. W is finite.
2. W is a finite reflection group.
3. The bilinear form $(\cdot, \cdot)$ on V is positive definite.

Bruhat order. The *Bruhat order* on W is the partial order $<$ generated by the relations $w < wt$ for $w \in W$ and $t \in T$ with $\ell(w) < \ell(wt)$. It holds that $u \leq v$ if and only if in each reduced expression for v there exists a subexpression equal to u. If $s \in S$ and $w \in W$ then $sw < w$ if and only if $\ell(sw) = \ell(w) - 1$, and $ws < w$ if and only if $\ell(ws) = \ell(w) - 1$.

1.3 Hecke algebras

Let (W, S) be a Coxeter system.

Generic algebra. Let A be a commutative ring. Choose elements $a_s, b_s \in A$ for each $s \in S$ such that $a_s = a_t$ and $b_s = b_t$ if $s, t \in S$ are conjugate in W. Let $\mathcal{H}$ be the free A-module with a basis given by $\{T_w : w \in W\}$. There exists a unique A-algebra structure on $\mathcal{H}$ with unit element $T_1 = 1$ and for all $s \in S$ and $w \in W$

$$\begin{cases} T_s^2 = a_s T_s + b_s \\ T_s T_w = a_s T_w + b_s T_{sw} & \text{if } sw < w \\ T_w T_s = a_s T_w + b_s T_{ws} & \text{if } ws < w \\ T_u T_v = T_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v). \end{cases}$$

This is the *generic (Hecke) algebra* of (W, S).

If A is a field K and $a_s = q - 1$ and $b_s = q$ where $\mathbb{F}_q$ is a finite field, then $\mathcal{H}$ is isomorphic to the algebra $eKGe \cong \mathrm{End}_{KG}(eKG)$ where $e = \frac{1}{|B|}\sum_{b \in B} b \in KG$ and G is a finite group of Lie type over $\mathbb{F}_q$ and $B \subset G$ is a Borel subgroup. Under some mild conditions, there is a natural correspondence between irreducible submodules of eKG and irreducible $\mathcal{H}$-modules, which transforms multiplicities to degrees.

Iwahori-Hecke algebra. This situation motivates us to pay especial attention to the following specialization of the generic algebra. From now on, let $A = \mathbb{Z}[x, x^{-1}]$ and $a_s = x^2 - 1$ and $b_s = x^2$ where x is an indeterminate. The corresponding algebra $\mathcal{H}$ is the *Iwahori-Hecke algebra* of (W, S).

Let $H_w = x^{-\ell(w)}T_w$ so that $\mathcal{H} = \mathbb{Z}[x,x^{-1}]\text{-span}\{H_w : w \in W\}$. If $s \in S$ and $w \in W$ then

$$\begin{cases} H_s^2 = 1 + (x - x^{-1})H_s \\ H_s H_w = H_{sw} + (x-x^{-1})Hw & \text{if } sw < w \\ H_w H_s = H_{ws} + (x-x^{-1})Hw & \text{if } ws < w \\ H_u H_v = H_{uv} & \text{if } \ell(uv) = \ell(u) + \ell(v). \end{cases}$$

It follows that $H_w = H_{s_1}H_{s_2}\cdots H_{s_k}$ if $w = s_1 s_2 \cdots s_k$ is any reduced expression for $w \in W$.

Bar involution. The *bar involution* of $\mathcal{H}$ is the unique ring automorphism $h \mapsto \overline{h}$ of $\mathcal{H}$ with $\overline{x} = x^{-1}$ and $\overline{H_s} = H_s^{-1}$ for $s \in S$ and $\overline{H_w} = (H_{w^{-1}})^{-1}$ for $w \in W$. By definition $\overline{gh} = \overline{g}\cdot\overline{h}$ and $\overline{g+h} + \overline{g} + \overline{h}$ for all $g,h \in \mathcal{H}$. It follows that $\overline{\overline{h}} = h$ for $h \in \mathcal{H}$, and $\overline{H_s} = H_s + x^{-1} - x$ for $s \in S$.

Kazhdan-Lusztig basis. If $w \in W$ then

$$\overline{H_w} \in H_w + \sum_{v<w} \mathbb{Z}[x,x^{-1}]H_v$$

where $<$ is the Bruhat order on W. For each $w \in W$ there exists a unique element $C_w \in \mathcal{H}$ with

$$\overline{C_w} = C_w \in H_w + \sum_{v<w} x^{-1}\mathbb{Z}[x^{-1}]H_v.$$

The set $\{C_w : w \in W\}$ is a $\mathbb{Z}[x,x^{-1}]$-basis for $\mathcal{H}$, called the *Kazhdan-Lusztig (KL) basis*.

We have $C_1 = 1$ and $C_s = H_s + x^{-1}$ for $s \in S$. Define $h_{yw} \in \mathbb{Z}[x^{-1}]$ such that $C_w = \sum_{y\in W} h_{yw}H_y$. Let $\mu(y,w)$ be the coefficient of x^{-1} is h_{yw} and set $P_{yw} = x^{\ell(w)-\ell(y)}h_{yw}$.

The importance of the KL basis has to do with the positivity properties of these polynomials, and the fact that their values $P_{yw}(1)$ at $x = 1$ encode the multiplicities of the irreducible submodules of Verma modules of a complex semisimple Lie algebra with Weyl group given by W.

If $w \in W$ and $s \in S$ then

$$C_s C_w = \begin{cases} (x + x^{-1})C_s & \text{if } sw < w \\ \\ C_{sw} + \displaystyle\sum_{\substack{y \in W \\ sy<y<w}} \mu(y,w)C_y & \text{if } sw > w. \end{cases}$$

There is an almost identical right-handed formula for $C_w C_s$. From this formula, it is elementary to prove by induction that each P_{yw} is an element of $\mathbb{Z}[x^2]$ with degree at most $\ell(w - \ell(y) - 1$ if $y < w$. It actually holds that $P_{yw} \in \mathbb{N}[x^2]$ but this is much more difficult to prove.

2 Left cells

Let R be a commutative ring.

Let A be an R-algebra which is free as an R-module with basis $\{b_w\}_{w\in W}$ indexed by some set W.

Suppose A is generated by some elements $\{g_s\}_{s\in S}$ indexed by a set S.

We have in mind the case when (W,S) is a Coxeter system and $A = \mathcal{H}$, but for the following constructions we do not need any of this extra structure.

For each $s \in S$ and $u, v \in W$ define $m_s(u \to v) \in A$ such that $g_s b_u = \sum_{v\in W} m_s(u \to v)b_v$.

Definition. The *left cell graph* of $(A, \{b_w\}_{w\in W}, \{g_s\}_{s\in S})$ is the direct graph with verte set W and an edge $u \to v$ if and only if $m_s(u \to v)$ is nonzero for some $s \in S$.

A subset $\mathscr{C} \subset W$ is a *left cell* if it is a strongly connected component of the left cell graph, i.e., if there exists a direct path from u to v and from v to u for any $u, v \in \mathscr{C}$.

If $\mathscr{C} \subset W$ is a left cell, define $\mathscr{C}^+$ as the set of $w \in W \setminus \mathscr{C}$ such that there exists a directed path from some (equivalently, every) $u \in \mathscr{C}$ to w in the left cell graph.

Proposition. If $I = R\text{-span}\{b_w : w \in \mathscr{C} \cup \mathscr{C}^+\}$ and $J = R\text{-span}\{b_w : w \in \mathscr{C}^+\}$ then both I and J are left ideals in A. The quotient I/J is a free R-module with basis $\tilde{b}_w = b_w + J$ for $w \in sC$, and is a left A-module satisfying

$$g_s\tilde{b}_u = \sum_{v\in\mathscr{C}} m_s(u \to v)\tilde{b}_v$$

for $s \in S$ and $u \in \mathscr{C}$. We refer to I/J as the *left cell module* of $\mathscr{C}$.

Proof. By construction, if $s \in S$ and $w \in \mathscr{C} \cup \mathscr{C}^+$ then $g_sb_w \in I$. If $w \in \mathscr{C}^+$ then $g_sb_w \in J$ since if $g_sb_w \in I \setminus J$ then there would exist a directed path in the left cell graph from w to some $u \in \mathscr{C}$, implying $w \in \mathscr{C}$.

The formula for $g_s\tilde{b}_u$ holds by the definition of $m_s(u \to v)$. □

Example. If we take $R = \mathbb{Z}[x, x^{-1}]$ and $A = \mathcal{H}$ and $\{b_w\} = \{H_w\}$ and $\{g_s\} = \{H_s\}$, then the left cells are not very interesting: in this case $m_s(w \to sw) = 1$ for all $s \in S$ and $w \in W$ so the left cell graph is essentially just the Cayley graph of W, so there is only one left cell, consisting of all of W.

The left cells become more interesting if we replace the standard basis $\{H_w\}_{w\in W}$ with the KL basis.

Definition. The *left cells* of a Coxeter system (W, S) with Iwahori-Hecke algebra $\mathcal{H}$ are the left cells defined with respect to the left cell graph of $(\mathcal{H}, \{C_w\}_{w\in W}, \{C_s\}_{s\in S})$.

Note that $\{C_s\}_{s\in S}$ does generate $\mathcal{H}$. Replacing this generating set by $\{H_s\}_{s\in S}$ or even $\{T_s\}_{s\in S}$ makes no difference on the resulting set of left cells.

We can describe the left cell graph for (W, S) more explicitly in terms of the leading coefficients $\mu(y, w)$.

Proposition. If $s \in S$ and $u, v \in W$ then

$$m_s(u \to v) = \begin{cases} x + x^{-1} & \text{if } u = v \text{ and } su < u \\ 1 = \mu(u, v) & \text{if } v = su > u \\ \mu(v, u) & \text{if } sv < v < u \text{ and } su < u. \end{cases}$$

Consequently, if $u \neq v$ then there exists an edge $u \to v$ in the left cell graph of (W, S) if and only if $\mu(v, u)$ or $\mu(u, v)$ is nonzero and there exists $s \in S$ with $sv < v$ but $u < su$.

Proof. This mostly follows from the formula for C_sC_w stated earlier.

We proved last time that if $u < v$ and $sv < v$ and $u < su$ then $P_{uv} = P_{su,v}$, and this implies that $\mu(u, v) = 1$ if $v = su > u$.

If $u < v$ and $sv > v$, $u < su$ then it follows similarly that $h_{uv} = x^{-1}h_{su,v}$ so if $su \neq v$ then $h_{uv} \in x^{-2}\mathbb{Z}[x^{-1}]$ and $\mu(u, v) = 0$.

Thus if $u \neq v$ then $m_s(u \to v)$ is nonzero if and only if $\mu(v, u)$ or $\mu(u, v)$ is nonzero and s is a left descent of v but not u. □

This lets us recover the description of the left cells from last time.

Let $y, w \in W$. Define $L(w) = \{s \in S : sw < w\}$ and $R(w) = \{s \in S : ws < w\}$.

Write $y \sim w$ if $\mu(y, w) \neq 0$ or $\mu(w, y) \neq 0$.

Write $y \leq_L w$ if there exists a chain $y = y_0 \sim y_1 \sim \cdots \sim y_r = w$ such that $L(y_i) \not\subset L(y_{i+1})$ for each i.

Corollary. We have $u \leq_L v$ if and only if there exists a directed path from v to $u$in the left cell graph.

Write $y \sim_L w$ if $y \leq_L w$ and $w \leq_L y$.

Corollary. Each left cell in W is an equivalence class under the transitive relation generated by $\sim_L$.

Example. Suppose $W = \langle s, t \rangle$ is a dihedral group of size $2m$. Then $P_{uv} = x^{\ell(u)-\ell(v)} h_{uv} = 1$ for all $u, v \in W$ with $\ell(u) < \ell(v)$ so $\mu(u, v) = 1$ if $\ell(u) = \ell(v) - 1$ and otherwise $\mu(u, v) = 0$. After drawing the left cell graph, one sees that there are four lefts cells, given by $\{1\}$, $\{w_0\}$, $\{a, ba, aba, \dots\}$ and $\{b, ab, bab, \dots\}$.