1 Last time: cells

Let (W, S) be a Coxeter system with Iwahori-Hecke algebra $\mathcal{H} = \mathbb{Z}[x, x^{-1}]\text{-span}\{H_w : w \in W\}$. Let

$$C_w = \overline{C_w} = \sum_{y \in W} h_{yw} H_y$$

be the KL basis element of $w \in W$. Here $h_{yw} \in x^{-1}\mathbb{Z}[x^{-1}]$ if $y < w$, $h_{ww} = 1$, and $h_{yw} = 0$ if $y \not\leq w$ where $\leq$ is the Bruhat order on W.

Let $\mu(y, w)$ be the coefficient of x^{-1} of h_{yw} for $y, w \in W$. Note that $\mu(y, w) \neq 0$ only if $y < w$.

The *left cell graph* of (W, S) is the directed graph with vertex set W in which an edge connects $u \to v$ if and only if C_v appears with nonzero coefficient in the expansion of the product $C_s C_u$ in the KL basis for some $s \in S$. (Recall that $C_s = H_s + x^{-1}$.)

The *left cells* of (W, S) are the sets of vertices in the strongly connected components of the left cell graph.

The *right cell graph* and *right cells* of (W, S) are defined similarly.

For $w \in W$ let $\mathrm{Des}_R(w) = \{s \in S : ws < w\}$ and $\mathrm{Des}_L(w) = \{s \in sw < w\}$.

Example. If $S = \{s, t\}$ so that $W = \langle s, t \rangle$ is a dihedral group of order $2m$ for $m = m(s, t) < \infty$, then there are 4 left cells, given by the subsets of elements in W with the same right descent set: $\{1\}$, $\{w_0\}$, $\{s, ts, sts, \dots\}$, and $\{t, st, tst, \dots\}$.

One motivation for considering left cells is as a means of constructing representations of $\mathcal{H}$:

Proposition. If $\mathscr{C} \subset W$ is a left cell, then the free $\mathbb{Z}[x, x^{-1}]$-module $M_{\mathscr{C}} = \mathbb{Z}[x, x^{-1}]\text{-span}\{m_w : w \in \mathscr{C}\}$ has a unique left $\mathcal{H}$-module structure in which

$$C_s m_w = \begin{cases} (x + x^{-1}) m_w & \text{if } sw < w \\ m_{sw} + \displaystyle\sum_{\substack{y \in \mathscr{C} \\ sy < y < w}} \mu(y, w) m_y & \text{if } sw > w \end{cases}$$

for $s \in S$ and $w \in \mathscr{C}$. Call this the *left cell module* of $\mathscr{C}$.

Special properties of the KL basis imply a simpler characterization of the left cells.

Definition. Let $y, w \in W$.

1. Write $y \sim w$ if $\mu(y, w) \neq 0$ or $\mu(w, y) \neq 0$.
2. Write $y \leq_L w$ if there are $y = y_0 \sim y_1 \sim \cdots \sim y_r = w$ with $\mathrm{Des}_L(y_i) \not\subset \mathrm{Des}_L(y_{i+1})$ for each i.
3. Write $y \sim_L w$ if $y \leq_L w$ and $w \leq_L y$.

Proposition. The left cells in W are the equivalence classes under the relation $\sim_L$.

We will need this lemma from a few lectures ago:

Lemma. Let $s \in S$ and $u, v \in W$ with $u < v$.

1. If $sv < v$ and $u < su$ then $h_{uv} = x^{-1} h_{su,v}$
2. If $vs < v$ and $u < us$ then $h_{uv} = x^{-1} h_{us,v}$.

In these cases, it follows that $\mu(u, v) \neq 0$ if and only if $su = v$ in (1) or $us = v$ in (2).

The phenomenon observed in our Example for a finite dihedral group has this generalization:

Proposition. Let $u, v \in W$. If $u \leq_L v$ then $\mathrm{Des}_R(u) \supset \mathrm{Des}_R(v)$, so if $u \sim_L v$ then $\mathrm{Des}_R(u) = \mathrm{Des}_R(v)$.

Proof. Suppose $u \leq_L v$. It suffices to assume that $u \sim v$, so that $\mathrm{Des}_L(u) \not\subset \mathrm{Des}_L(v)$.

If $u < v$ then $\mu(v,u) = 0$ so we must have $\mu(u,v) \neq 0$. It follows by the lemma that every right descent of v must also be a right descent of u.

If $v < u$ then $\mu(u,v) = 0$ so we must have $\mu(v,u) \neq 0$. Since there exists $s \in S$ with $su < u$ and $v < sv$ it follows by the lemma that $v < sv = u$, from which it is clear that $\mathrm{Des}_R(v) \subset \mathrm{Des}_R(u)$. □

2 Left cells in type A

We specialize to the case when $W = S_n$ and $S = \{s_i = (i, i+1) : i = 1, 2, \dots, n-1\}$.

Definition. Say that two elements $u, v \in S_n$ are *dual Knuth equivalent*, and write $u \overset{i}{\underset{dK}{\approx}} v$, if $su < u < s'u = v < sv$ where $\{s, s'\} = \{s_{i-1}, s_i\}$ for some $1 < i < n$.

Proposition. If $u, v \in S_n$ then $u \overset{i}{\underset{dK}{\approx}} v$ if and only if there are numbers $1 \leq a < b < c \leq n$ such that $(u^{-1}(i-1)u^{-1}(i)u^{-1}(i+1), v^{-1}(i-1)v^{-1}(i)v^{-1}(i+1))$ is (bac, bca) or (acb, cab). Here we write "xyz" not to denote the product of three numbers, but as a shorthand for the triple (x, y, z).

The preceding property is straightforward to check from the definition of $\overset{i}{\underset{dK}{\approx}}$. The details are left to the reader.

Lemma. If $u, v \in S_n$ then $u \overset{i}{\underset{dK}{\approx}} v$ for some i if and only if there are antiparallel edges $u \to v$ and $u \leftarrow v$ in the left cell graph of S_n. Therefore if $u \overset{i}{\underset{dK}{\approx}} v$ then $u \sim_L v$.

Proof. Suppose $su < u < s'u = v < sv$ for $\{s, s'\} = \{s_{i-1}, s_i\}$. Then $C_{s'}C_u = C_v + (\text{ other terms })$ so there exists an edge $u \to v$. We also have $\mu(u,v) = 1$ since $h_{uv} = x^{-1}h_{s'u,v} = x^{-1}h_{vv} = x^{-1}$ so $C_sC_v = C_{sv} + C_u + (\text{ other terms })$.

Conversely, assume $\ell(u) < \ell(v)$ and there are edges $u \to v$ and $u \leftarrow v$ in the left cell graph of S_n. We can only have an edge $u \to v$ if $u < s'u = v$ for some $s' \in S$. In this case, there can only be an edge $u \leftarrow v$ if for some $s \in S$ we have $su < u < s'u = v < sv$; note that in this case $\mu(u,v) = 1$ holds automatically. This, finally, can only occur if s and s' do not commute, so $\{s, s'\} = \{s_{i-1}, s_i\}$ for some i. □

The *RSK correspondence* is (for our purposes) as bijection from S_n to the set of pairs (P, Q) of standard tableaux of the same shape λ, where λ is an integer partition of n.

Here, a *tableau* T of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0)$ is a map $D_\lambda \to \{1, 2, 3, \dots\}$, where

$$D_\lambda = \{(i,j) : 1 \leq j \leq \lambda_i,\ 1 \leq i \leq k\}.$$

We identify D_λ with a subset of positions in a matrix; then T corresponds to a way of filling these positions with numbers.

For example, if $\lambda = (3, 2, 1)$ then every T of shape λ has the form $T = \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}$

As tableau is *standard* if its entries are increasing from left to right in each row and from top to bottom in each column, and the entries which occur form a consecutive sets of integers $\{1, 2, 3, \dots, n\}$.

Standard: $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

Not standard: $\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$

Bumping Algorithm. Given a row $\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_k \\ \hline\end{array}$ and a number j, define

$$\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_k \\ \hline\end{array} \leftarrow j$$

as the new row given by locating the first entry from the left with $j < i_r$ and replacing i_r by j, or by adding j to the end of the row if no such i_r exists. In the first case, we say that i_r is *bumped*.

For example,

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array} \leftarrow 5 = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array}$$

(no entry is bumped) and

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array} \leftarrow 2 = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 4 \\ \hline\end{array}$$

(3 is bumped).

RSK algorithm. Given a permutation $w = w_1 w_2 \cdots w_n \in S_n$ where $w_i = w(i)$, start with the pair of empty tableaux $(P_0, Q_0) = (\varnothing, \varnothing)$, and for $i = 1, 2, \dots, n$ define the tableau P_i by inserting w_i into the first row of P_{i-1}, then inserting the bumped entry (if any exists) into the second row, then inserting the next bumped entry (if any exists) into the third row, and so on until some number is placed at the end of a row. Form Q_i by adding a box with i to Q_{i-1} in the location of the new entry in P_i. Each pair (P_i, Q_i) consists of two standard tableaux of the same shape, and we set $(P, Q) = (P_n, Q_n)$.

For example, if $w = 4213 \in S_4$ then we have

P_i	Q_i	i
$\varnothing$	$\varnothing$	0
$\begin{array}{|c|}\hline 4 \\ \hline\end{array}$	$\begin{array}{|c|}\hline 1 \\ \hline\end{array}$	1
$\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline\end{array}$	$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$	2
$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline\end{array}$	$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$	3
$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$	$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$	4

The following is well-known, so we won't prove it.

Theorem. The map $w \xmapsto{\text{RSK}} (P, Q)$ is a bijection from S_n to pairs of standard tableaux of the same shape with entries in $1, 2, \dots, n$.

Call $P = P(w)$ the *insertion tableau* and $Q = Q(w)$ the *recording tableau*.

Fact. Two permutations have the same recording tableau Q if and only if they are dual Knuth equivalent, that is, connected by a sequence of dual Knuth equivalences $\overset{i}{\underset{dK}{\approx}}$ (where i can vary).

We also won't prove this well-known property.

We sketch a proof of the following, however:

Theorem (Kazhdan and Lusztig (1979))**.** The dual Knuth equivalence classes in S_n are the left cells; i.e., each left cell in S_n has the form $\{w \in S_n : Q(w) = T\}$ for some standard tableau T.

Proof sketch. Consider the *Knuth equivalences* $\overset{i}{\underset{K}{\approx}}$ defined as the right-handed version of $\overset{i}{\underset{dK}{\approx}}$, i.e., by setting $u \overset{i}{\underset{K}{\approx}} v$ if $us < u < us' = v < vs$ for some $\{s, s'\} = \{s_{i-1}, s_i\}$. Note that this relation does not imply $\sim_L$. (Rather, by symmetric arguments it follows that $u \overset{i}{\underset{K}{\approx}} v$ implies the right cell relation $u \sim_R v$, though this property is of no consequence to our proof.)

If $u \in S_n$ and exactly one of s_{i-1} or s_i is a right descent of u, then one can show that there exists a unique permutation $u^* \overset{i}{\underset{K}{\approx}} u$, and in this case if $u \sim_L v$ then it also holds that exactly one of s_{i-1} or s_i is a right descent of v, and we have $u^* \sim_L v^*$. Thus, one can "transport" a left cell equivalence $u \sim_L v$ to another equivalence $u^* \sim_L v^*$ in a different left cell.

One can also show that $Q(u)$ is uniquely determined by i and $Q(u^*)$, where i is such that $u \overset{i}{\underset{K}{\approx}} u^*$.

Now suppose $u \sim_L v$. Write $y \underset{K}{\approx} z$ if y and z are connected by a sequence of equivalences $\overset{i}{\underset{K}{\approx}}$.

We claim that $\left\{\mathrm{Des}_R(z) : z \underset{K}{\approx} u\right\} = \left\{\mathrm{Des}_R(z') : z' \underset{K}{\approx} v\right\}$. This holds since whenever there is a path

$$u = u_0 \overset{i_1}{\underset{K}{\approx}} u_1 \overset{i_2}{\underset{K}{\approx}} u_2 \overset{i_3}{\underset{K}{\approx}} \cdots \overset{i_l}{\underset{K}{\approx}} u_l = z$$

then $u_i = u^*_{i-1}$ for all i, so there is also a path

$$v = v_0 \overset{i_1}{\underset{K}{\approx}} v_1 \overset{i_2}{\underset{K}{\approx}} v_2 \overset{i_3}{\underset{K}{\approx}} \cdots \overset{i_l}{\underset{K}{\approx}} v_l = z'$$

where $v_i = v^*_{i-1}$, and it holds that $u_i \sim_L v_i$ for all i, so $\mathrm{Des}_R(u_i) = \mathrm{Des}_R(v_i)$ and $\mathrm{Des}_R(z) = \mathrm{Des}_R(z')$.

Now, suppose z is such that $\mathrm{Des}_R(z)$ is the lexicographically last set in $\left\{\mathrm{Des}_R(z) : z \underset{K}{\approx} u\right\}$. One can show that $Q(z)$ is *row superstandard* (i.e., each of its rows consists of consecutive numbers $i, i+1, i+2, \dots$) and uniquely determined by $\mathrm{Des}_R(z)$. It follows in this case that z', which has the same descent set as z, must also have row superstandard recording tableau $Q(z') = Q(z)$

Since each $Q(u_j)$ is uniquely determined by i_j and $Q(u_{j+1})$, it follows that $Q(u_j) = Q(v_j)$ for all j, so $Q(u) = Q(v)$. This suffices to prove the theorem. □