1 Positivity properties of the Kazhdan-Lusztig basis

Let (W, S) be a Coxeter system with Bruhat order $<$ and length function ℓ.

Let $\mathcal{H} = \mathbb{Z}[x, x^{-1}]\text{-span}\{H_w : w \in W\}$ be the Iwahori-Hecke algebra of (W, S), the unique $\mathbb{Z}[x, x^{-1}]$-algebra with $H_s^2 = 1 + (x - x^{-1})H_s$ for $s \in S$ and $H_u H_v = H_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$.

Define $h \mapsto \overline{h}$ as the unique ring involution of $\mathcal{H}$ with $\overline{x^n} = x^{-n}$ and $\overline{H_w} = (H_{w^{-1}})^{-1}$. Recall that the KL basis $\{C_w\}_{w \in W}$ of $\mathcal{H}$ is defined as the unique set of elements with $C_w = \overline{C_w} \in H_w + \sum_{y<w} x^{-1}\mathbb{Z}[x^{-1}]H_y$.

Theorem (Elias and Williamson)**.** The KL basis has the following positivity properties:

1. $C_w \in \mathbb{N}[x^{-1}]\text{-span}\{H_y : y \in W\}$.
2. $C_u C_v \in \mathbb{N}[x, x^{-1}]\text{-span}\{C_w : w \in W\}$.

Our goal in the last two lectures is to sketch a heuristic explanation for these properties, which seem to have no elementary proof. The main idea is that $\mathcal{H}$ and its KL basis can be "categorified," meaning informally that we can define a category $\mathscr{C}$ for where there are correspondences

$\mathcal{H} \longleftrightarrow [\mathscr{C}]$, the "split Grothendieck group" of $\mathscr{C}$

bar involution $\longleftrightarrow$ a duality functor on $\mathscr{C}$

KL basis $\longleftrightarrow$ isomorphism classes of indecomposable objects in $\mathscr{C}$

multiplication by x^n $\longleftrightarrow$ shift an object's grading by n

addition $\longleftrightarrow \oplus$

multiplication $\longleftrightarrow \otimes$

Some standard definitions from category theory are in order to make this picture (slightly more) rigorous.

A *category* $\mathscr{C}$ consists of a collection of objects and for each pair of objects a set of morphisms from one to the other. If $\text{Mor}(A, B)$ is the set of morphisms $A \to B$, then we can compose morphisms $f : A \to B$ and $g : B \to C$ to get a morphism $g \circ f \in \text{Mor}(A, C)$. This composition must be associative, and for each object A there must exist an identity morphism $\text{id}_A \in \text{Mor}(A, A)$ which, when composed with a morphism on the left or the right, yields back the morphism unchanged.

A *subcategory* $\mathscr{D}$ of $\mathscr{C}$ is a category in which every object is also an object of $\mathscr{C}$, and $\text{Mor}_{\mathscr{D}}(A, B) \subseteq \text{Mor}_{\mathscr{C}}(A, B)$. The subcategory is *full* if the containment of morphisms is always equality.

Example. The category *Ab* of abelian groups: objects are abelian groups and morphisms are group homomorphisms. This is a full subcategory of the category of all groups.

Example. The category of (left) modules over a commutative ring R: morphisms are R-module homomorphisms.

Example. The category R-Bim of R-bimodules where R is a commutative ring: objects are abelian groups M which are simultaneously left and right R-modules, such that $(rm)s = r(ms)$ for all $m \in M$ and $r, s \in R$. This compatibility condition means we can write rms without ambiguity. The morphisms are abelian group homomorphisms $\phi : M \to M'$ with $\phi(rms) = r\phi(m)s$ for $m \in M$ and $r, s \in R$.

All of these categories, and all categories we will work with are subcategories (usually not full) of *Ab*, though many constructions we'll see could be considered in a more abstract setting.

The category *Ab* is *additive*: it contains the direct sum $A \oplus B$ of any two objects and there exists a 0-object such that $A \oplus 0 \cong 0 \oplus A \cong A$ for any object A. The notion of an *additive category* is defined in general by the same properties (once we say how direct sums are specified by a universal property).

The category R-Bim is also additive.

An object in an additive category is *indecomposable* if it is not a direct sum of two nonzero objects.

Definition. The *split Grothendieck group* $[\mathscr{C}]$ of an additive category $\mathscr{C}$ is the abelian group generated by the symbols $[M]$ for objects M in $\mathscr{C}$, subject to the relations $[M] = [A] + [B]$ if $M \cong A \oplus B$.

There is a related construction, called the *Grothendieck group*, which is defined for any *abelian category*. Note that $[M] = [M] + [0]$ for all M so $[0] = 0$ in $[\mathscr{C}]$, and if $M \cong N$ then $[M] = [N]$ since $M \cong N \oplus 0$. The group $[\mathscr{C}]$ is abelian since it always holds in an additive category that $M \oplus N \cong N \oplus M$.

Example. If $\mathscr{C}$ is the full subcategory of R-modules which are free and finitely generated (which is the category of finite-dimensional vector spaces if R is a field) then $\mathscr{C}$ is additive and $[\mathscr{C}] \cong \mathbb{Z}$.

If $\mathscr{C}$ is a *monoidal category*, meaning there exists a notion of a tensor product $\otimes$ and a unit object $\mathbf{1}$ satisfying certain associativity axioms, then $[\mathscr{C}]$ is a ring with respect to the multiplication $[A][B] = [A \otimes B]$ and unit $[\mathbf{1}]$.

The category R-Bim, where R is a fixed commutative ring, is additive and monoidal. The tensor product $M \otimes N$ of two R-bimodules M and N is defined as the quotient of the abelian group $M \times N = \{(m, n) : m \in M,\ n \in N\}$ by the relations $(m_1 + m_2, n) = (m_1, n) + (m_2, n)$ and $(m, n_1 + n_2) = (m, n_1) + (m, n_2)$ and $r(m, n) = (rm, n) = (m, rn)$ and $(m, n)r = (m, nr) = (mr, n)$ for $m, m_i \in M$ and $n, n_i \in N$ and $r \in R$. (There is a more concise, but less constructive definition if you know the universal property of the tensor product.) One denotes the equivalence class of (m, n) in $M \otimes N$ by $m \otimes_R n$. Note that R is itself an R-bimodule (which serves the unit object with respect to $\otimes$) and that $R \otimes M \cong M \otimes R \cong M$ as bimodules.

A *graded R-bimodule* is an R-bimodule M with a direct sum decomposition $M = \bigoplus_{i \in \mathbb{Z}} M^i$. Here, each M^i is an R-bimodule, called the ith graded piece of M, whose elements are said to have *degree* i. (Not every element of M has a well-defined degree—only the elements which belong to M^i for some i.)

The direct sum of two graded R-bimodules M and N is graded: $M \oplus N = \bigoplus_{i \in \mathbb{Z}} M^i \oplus N^i$.

The tensor product is also graded, in a slightly less straightforward way: $M \otimes N = \bigoplus_{i \in \mathbb{Z}} (M \otimes N)^i$ where $(M \otimes N)^i = \bigoplus_{j+k=i} M^j \otimes N^k$. Thus if $m \in M^j$ and $n \in N^k$ then $m \otimes_R n$ has degree $j + k$.

We consider R to be graded with $R^0 = R$ and $R^i = 0$ for $i \neq 0$.

A *graded morphism* of graded R-bimodules $\phi : M \to N$ is an R-bimodule homomorphism with $\phi(M^i) \subset N^i$ for all $i \in \mathbb{Z}$.

The category of graded R-bimodules with graded morphisms is an additive and monoidal, but not full, subcategory of R-Bim. The split Grothendieck group of this subcategory, and more generally of any additive category whose objects and morphisms are graded, is naturally a $\mathbb{Z}[x, x^{-1}]$-algebra: the group is already a ring, and becomes a $\mathbb{Z}[x, x^{-1}]$-module on setting $(\sum_{n \in \mathbb{Z}} a_n x^n)[M] = \left[\bigoplus_{n \in \mathbb{Z}} M(n)^{\oplus a_n}\right]$. Here $M(n)$ is the graded R-bimodule with $M(n)^i = M^{i+n}$, i.e., given by shifting the grading on M by n.

Example (Soergel bimodules for S_1)**.** Any left R-module can be considered as an R-bimodule on which the right action of R is trivial. Via this identification, we get analogous notions of graded left R-modules and graded left R-module morphisms, as well as direct sums and tensor products of these objects. Suppose $\mathscr{C}$ is the additive, monoidal category whose objects are the graded left R-modules which are finitely generated and free, with graded morphisms.

As a $\mathbb{Z}[x, x^{-1}]$-algebra, we have $[\mathscr{C}] \cong \mathbb{Z}[x, x^{-1}]$. Each object is isomorphic to a sum of grading shifts of the graded R-module given by R itself (whose elements are all in degree 0). The map $[R] \mapsto 1$ extends to an algebra isomorphism.

Given an object M is $\mathscr{C}$, define $\overline{M}$ as the graded R-module whose jth graded piece is M^{-j}. Then $[\overline{M}] = \overline{[M]}$ where $h \mapsto \overline{h}$ is the usual bar involution of $\mathbb{Z}[x, x^{-1}] = \mathcal{H}$ for $W = S_1$.

2 Soergel bimodules for S_2

Our goal is to define a full subcategory of the category of graded R-bimodules, whose split Grothendieck group is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}$ of an arbitrary Coxeter group. We will call this the category $\mathbb{S}$Bim of *Soergel bimodules*.

The precise definition will come next time. In this lecture, we confine ourself to a concrete description of $\mathbb{S}$Bim in the simplest nontrivial but highly instructive case when $W = S_2$.

Let $W = S_2$ be the unique Coxeter group with one generator, so that $S = \{s = (1, 2)\}$.

Define $R = \mathbb{R}[x]$ as the ring of real polynomials in one variable. Usually this ring is graded with $R^i = \mathbb{R}x^i$, but for technical reasons we adopt the perverse convention that x^i has degree $2i$.

W acts on R by $(s \cdot f)(x) = f(-x)$. For example, $s \cdot (1 + x + x^2 + x^3) = 1 - x + x^2 - x^3$.

Let $R^s = \{f \in R : sf = f\} = \mathbb{R}[x^2]$.

For $k \in \mathbb{N}$, define the R-bimodules

$B_1 = R$.

$B_s = R \otimes_{R^s} R(1)$.

$B_{[k]} = \underbrace{R \otimes_{R^s} R \otimes_{R^s} \cdots \otimes_{R^s} R}_{k+1 \text{ factors}}(k)$.

Here B_s is the tensor product of R with itself, but regarded as an R^s-bimodule, with its grading shifted down by 1. This means that elements of B_s are sums of tensors of the form $af \otimes_{R^s} b = a \otimes_{R^s} fb$ where $a, b \in R$ and $f \in R^s$. If a and b have degrees j and k in R, then $a \otimes_{R^s} b$ has degree $j + k - 1$. Likewise, $B_{[k]}$ is the tensor product over R^s of $k + 1$ copies of R, with the grading shifted down by k.

$1 \in B_1$ has degree 0.

$1 \otimes_{R^s} 1 \in B_s$ has degree -1.

$1 \otimes_{R^s} \cdots \otimes_{R^s} 1 \in B_{[k]}$ has degree $-k$.

$x \otimes_{R^s} x^2 = x^3 \otimes_{R^s} 1 \in B_s$ has degree $2 + 4 - 1 = 5$.

Note that $B_1 = B_{[0]}$ and $B_s = B_{[1]}$. Moreover, we have

$$B_{[k]} \cong \underbrace{B_s \otimes B_s \otimes \cdots \otimes B_s}_{k \text{ factors}}$$

where here $\otimes$ is over R, distinct from $\otimes_{R^s}$.

Claim. The following holds:

(1) Both B_1 and B_s are indecomposable R-bimodules.

(2) $B_{[2]} \cong B_s(1) \oplus B_s(-1)$.

Proof. (1) Both B_1 and B_s are generated as R-bimodules by the single elements 1 and $1 \otimes_{R^s} 1$ of lowest degree, so must be indecomposable. (2) $B_{[2]}$ is the direct sum of sub-R-bimodules generated by

$$\alpha = 1 \otimes_{R^s} 1 \otimes_{R^s} 1 \qquad \text{and} \qquad \beta = 1 \otimes_{R^s} x \otimes_{R^s} 1.$$

Note that α has degree -2 and β has degree 0. One checks that $\langle \alpha \rangle \cong B_s(1)$ and $\langle \beta \rangle \cong B_s(-1)$. □

Let $\mathbb{S}$Bim be the full subcategory of the category of graded R-bimodules which contains all direct sums and grading shifts of the bimodules $B_{[k]}$ for $k \in \mathbb{N}$.

Proposition. The following holds for the category $\mathbb{S}\mathrm{Bim}$:

(a) $\mathbb{S}\mathrm{Bim}$ is closed under $\otimes$ so $[\mathbb{S}\mathrm{Bim}]$ is a $\mathbb{Z}[x, x^{-1}]$-algebra.

(b) Each indecomposable object in $\mathbb{S}\mathrm{Bim}$ is isomorphic to some grading shift of one of B_1 or B_s.

(c) There exists a unique algebra isomorphism $\varepsilon : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}]$ with $C_w \mapsto [B_w]$ for $w \in S_2$.

Proof. (a) This holds as $B_{[k]} \otimes B_{[l]} \cong B_{[k+l]}$. (b) This follows from the claim. Note that B_1 and B_s are not isomorphic to any grading shifts of each other. (c) Recall that $C_1 = 1$ and $C_s = H_s + x^{-1}$. Clearly $C_1 C_2 = C_w$ and $[B_1][B_w] = [B_1 \otimes B_w] = [B_w]$ for all $w \in S_2$. We have $C_s^2 = (x + x^{-1})C_s$ and $[B_s][B_s] = [B_s \otimes B_s] = [B_s(1) \oplus B_s(-1)] = (x + x^{-1})[B_s]$. The result follows. □

We have thus shown for $W = S_2$ that there exists a category of graded R-bimodules $\mathbb{S}\mathrm{Bim}$ such that $\mathcal{H} \cong [\mathbb{S}\mathrm{Bim}]$ in whih the KL basis elements correspond to a complete list of indecomposable objects up to grading shift and isomorphism. The (trivial) claim that $C_u C_v \in \mathbb{N}[x, x^{-1}]\text{-span}\{C_w : w \in S_2\}$ for $u, v \in S_2$ follows immediately, since $\varepsilon(C_u C_v) = \varepsilon(C_u)\varepsilon(C_v) = [B_u \oplus B_v]$.

Next time we will sketch how to extend this construction from $W = S_2$ to an arbitrary Coxeter group W. The key changes will be to define R as the ring of polynomial functions on the geometric representation of W. The definitions of R^s, B_s, and $\mathbb{S}\mathrm{Bim}$ will be similar.