1 Outline

Let (W, S) be a Coxeter system with Iwahori-Hecke algebra $\mathcal{H} = \mathbb{Z}[x, x^{-1}]\text{-span}\{H_w : w \in W\}$ and Kazhdan-Lusztig basis $\{C_w : w \in W\}$. Our goal in today's final lecture is to provide an explanation for the following positivity properties:

Theorem (Elias and Williamson (2012))**.** For all $u, v, w \in W$:

1. $C_w \in \mathbb{N}[x^{-1}]\text{-span}\{H_y : y \in W\}$.
2. $C_u C_v \in \mathbb{N}[x, x^{-1}]\text{-span}\{C_y : y \in W\}$.

Let $\mathscr{C}$ be a category. If $\mathscr{C}$ is additive (informally, closed under finite direct sums and containing a 0-object), then the *split Grothendieck group* $[\mathscr{C}]$ is the abelian group generated by $[M]$ for all objects M in $\mathscr{C}$, subject to the relations $[M] = [A] + [B]$ if $M \cong A \oplus B$.

We saw last time that if $\mathscr{C}$ is monoidal (informally, closed under finite tensor products and having a unit object $\mathbf{1}$), then $[\mathscr{C}]$ is naturally a ring, with unit $[\mathbf{1}]$ and multiplication $[A][B] = [A \otimes B]$.

If furthermore the objects and morphisms in $\mathscr{C}$ are $\mathbb{Z}$-graded, then $[\mathscr{C}]$ is a $\mathbb{Z}[x, x^{-1}]$-algebra: we define scalar multiplication by x^n as $x^n[M] = [M(n)]$ where $M(n)$ is the object given by shifting the grading of M down by n.

Soergel's key innovation in trying to prove the positivity properties of the KL basis was to introduce an additive, monoidal, and graded category $\mathbb{S}\text{Bim}$ whose split Grothendieck group is $\mathcal{H}$ and whose indecomposable objects correspond to $\{C_w : w \in W\}$.

We will locate $\mathbb{S}\text{Bim}$ as a full subcategory of the category of graded R-bimodules where R is a fixed commutative ring: recall from last time that there is a natural notion of direct sum, tensor product, and grading for R-bimodules and the homomorphisms between them.

2 Soergel bimodules

To define $\mathbb{S}\text{Bim}$ for an arbitrary Coxeter system (W, S), we first need to define the ring R.

Let $V = \mathbb{R}\text{-span}\{\alpha_s : s \in S\}$ be the familiar geometric representation of W.

Define R as the graded ring of polynomial functions on V, but grade the elements of R so that constant functions have degree 0, linear functions have degree 2 (rather than 1), quadratic functions have degree 4 (rather than 2), and so on.

We can think of R as a polynomial ring $R = \mathbb{R}[x_s : s \in S]$ in a commuting set of indeterminates x_s indexed by $s \in S$; here x_s acts as the linear function $V \to \mathbb{R}$ given by $x_s(\alpha_t) = \delta_{st}$.

In other words, R is the symmetric algebra on the dual space V^*.

Each $f \in R$ is a map $f : V \to \mathbb{R}$ and $w \in W$ acts on f by $(wf)(v) = f(w^{-1}v)$ for $v \in V$.

Let $R^s = \{f \in R : sf = f\}$ for $s \in S$.

Note that R is itself a graded R-bimodule (with all elements in degree 0).

Definition. For a finite sequence $\alpha = (s_1, s_2, \dots, s_k)$ with $s_i \in S$, define the graded R-bimodule B_α by

$$B_\alpha = \underbrace{R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R}_{k+1 \text{ factors}}(k).$$

Thus, we form a modified tensor product of $k+1$ copies of R, then shift the grading down by k so that $1 \otimes_{R^{s_1}} 1 \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} 1$ has degree $-k$.

Remark. Recall the definition of $\otimes_{R^s}$ from last time; we have

$$gk \otimes_{R^s} h = g \otimes_{R^s} kh \qquad \text{for all } g, h \in R,\ k \in R^s.$$

We view elements of B_α as sums of sequences

$$(f_0 \mid_{s_1} f_1 \mid_{s_2} f_2 \mid_{s_3} \cdots \mid_{s_k} f_k)$$

where each $f_i \in R$ and you can slide a scalar in R across the barrier $|_{s_i}$ if and only if the scalar is s_i-invariant.

Define $B_s = R \otimes_{R^s} R(1)$. For any $\alpha = (s_1, s_2, \dots, s_k)$, we then have

$$B_\alpha \cong B_{s_1} \otimes B_{s_2} \otimes \cdots \otimes B_{s_k}.$$

The tensor product $\otimes$ here is the usual one for R-bimodules, not the modified tensor product $\otimes_{R^s}$.

An object M is an additive category $\mathscr{C}$ is a *direct summand* of an object B if there exists an object N in $\mathscr{C}$ such that $B \cong M \oplus N$.

Definition. The category $\mathbb{S}$Bim of *Soergel bimodules* for a Coxeter system (W, S) is the smallest full subcategory of the category of graded R-bimodules which contains all direct summands of B_α for arbitrary sequences of simpler generators $\alpha = (s_1, s_2, \dots, s_k)$, and which is closed under finite direct sums and grading shifts.

Concretely, we form $\mathbb{S}$Bim by first taking all direct summands of the R-bimodules B_α, then including all finite direct sums of grading shifts of these bimodules.

Remark. The definition we saw last time for $\mathbb{S}$Bim when $W = S_2$ was slightly simpler than this one, since when $W = S_2$ every direct summand of B_α is a direct sum of grading shifts of the R-bimodules B_s.

3 Categorification theorems and Soergel's conjecture

An object in an additive category is *indecomposable* if it is not isomorphism to the direct sum of any two nonzero objects. By construction, the indecomposable objects of $\mathbb{S}$Bim are necessarily grading shifts of direct summands of the bimodules B_α.

Theorem (Soergel's Categorification Theorem I)**.** There is a unique isomorphism of $\mathbb{Z}[x, x^{-1}]$-algebras

$$\varepsilon : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}]$$

with $\varepsilon(C_s) = [B_s]$ for $s \in S$.

The uniqueness of such an isomorphism follows since the elements $\{C_s : s \in S\}$ generate $\mathcal{H}$. Checking that the map $C_s \mapsto [B_s]$ extends to a homomorphisms corresponds to checking certain isomorphisms in $\mathbb{S}$Bim associated to each braid relation in $\mathcal{H}$. Showing that ε is an isomorphism will follow from the next theorem which classifies the indecomposable objects in $\mathbb{S}$Bim.

Example (Soergel bimodules for $W = S_3$)**.** Suppose $W = S_3 = \{1, s, t, st, ts, sts = tst\}$.

We then have $S = \{s = (1, 2),\ t = (2, 3)\}$.

We may identify $R = \mathbb{R}[x, y, z]$, graded so that $x^i y^j z^k$ has degree $2(i + j + k)$.

W acts on $\mathbb{R}$ by $(s \cdot f)(x, y, z) = f(y, x, z)$ and $(t \cdot f)(x, y, z) = f(x, z, y)$.

The following are indecomposable Soergel bimodules:

$B_1 = R = \langle 1 \rangle$.

$B_s = \langle 1 \otimes_{R^s} 1 \rangle$.

$B_t = \langle 1 \otimes_{R^t} 1 \rangle$.

$B_{st} \overset{\text{def}}{=} B_s \otimes B_t = \langle 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \rangle$.

$B_{ts} \overset{\text{def}}{=} B_t \otimes B_s = \langle 1 \otimes_{R^t} 1 \otimes_{R^s} 1 \rangle$.

If $\delta = y - z$ and $\Delta = \delta \otimes_{R^t} 1 + 1 \otimes_{R^t} \delta$ then

$$B_s \otimes B_t \otimes B_s = \underbrace{\langle 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1 \rangle}_{\text{call this } B_{sts}} \oplus \underbrace{\rangle 1 \otimes_{R^s} \Delta \otimes_{R^t} 1 \rangle}_{\cong B_s}.$$

Likewise, if $\delta' = x - y$ and $\Delta' = \delta' \otimes_{R^s} 1 + 1 \otimes_{R^s} \delta'$ then

$$B_t \otimes B_s \otimes B_t = \underbrace{\langle 1 \otimes_{R^t} 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \rangle}_{\text{call this } B_{tst}} \oplus \underbrace{\rangle 1 \otimes_{R^t} \Delta' \otimes_{R^t} 1 \rangle}_{\cong B_t}.$$

Finally, one can show that $B_{sts} \cong B_{tst}$ and this bimodule is indecomposable.

Proposition. Each indecomposable object of $\mathbb{S}\mathrm{Bim}$ for $W = S_3$ is isomorphic to a grading shift of exactly one of the bimodules B_1, B_s, B_t, B_{st}, B_{ts}, or B_{sts}. Moreover, in this case $\varepsilon(C_w) = B_w$ for $w \in S_3$.

This example generalizes as follows:

Theorem (Soergel's Categorification Theorem II)**.** For each $w \in W$ there exists up to isomorphism a unique indecomposable bimodule B_w which occurs as a direct summand of B_α for any sequence $\alpha = (s_1, s_2, \dots, s_k)$ such that $w = s_1 s_2 \cdots s_k$ is a reduced expression, and which does not occur as a direct summand of $B_{\alpha'}$ for any shorter sequence α'. The resulting set of bimodules $\{B_w : w \in W\}$ represents all indecomposable objects in $\mathbb{S}\mathrm{Bim}$ up to $\cong$ and grading shift.

Our example shows that the following holds for $W = S_3$, but the statement is far from obvious in general.

Soergel's Conjecture. For all $w \in W$ it holds that $\varepsilon(C_w) = B_w$.

Elias and Williamson proved this conjecture, which immediately implies $C_u C_v \in \mathbb{N}[x, x^{-1}]$-span$\{C_w : w \in W\}$ for $u, v \in W$ since $C_u C_v = \varepsilon^{-1}(\varepsilon(C_u C_v)) = \varepsilon^{-1}([B_u \otimes B_v])$. To derive the other highlighted positivity property the KL basis, we need to recall an explicit formula which Soergel gave for ε^{-1}.

If M is a graded bimodule and $f = \sum_{n \in \mathbb{Z}} a_n x^n \in \mathbb{N}[x, x^{-1}]$ then let $M^{\oplus f} = \bigoplus_{n \in \mathbb{Z}} \underbrace{M(n) \oplus M(n) \oplus \cdots \oplus M(n)}_{a_n \text{ factors}}$.

Theorem (Soergel's Categorification Theorem III)**.** Every object M in $\mathbb{S}\mathrm{Bim}$ has a *standard filtration*, defined as the unique filtration of the form

$$0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(m)} = M$$

where $M^{(i)}/M^{(i-1)} \cong (R_{y_i})^{\oplus h_{y_i}}$ for some $y_i \in W$ and $h_{y_i} \in \mathbb{N}[x, x^{-1}]$ (where R_y is a "standard bimodule" whose definition we omit), such that $y_i < y_j$ in Bruhat order whenever $i < j$. Moreover, the map

$$\mathrm{ch} : [\mathbb{S}\mathrm{Bim}] \to \mathcal{H}$$

with $[M] \mapsto \sum_{i=1}^m h_{y_i} x^{\ell(y_i)} H_{y_i} \in \mathbb{N}[x, x^{-1}]$-span$\{H_y : y \in W\}$ is the inverse of $\varepsilon : \mathcal{H} \to [\mathbb{S}\mathrm{Bim}]$.

Since Soergel's conjecture implies that $\mathrm{ch}([B_w]) = C_w$, it follows that $C_w \in \mathbb{N}[x, x^{-1}]$-span$\{H_y\}$.

This concludes our course!

For further reading see: Libedinsky's Gentle introduction to Soergel bimodules and its references.