**FINAL EXAMINATION SOLUTIONS - MATH 2121, FALL 2017.**

Name:

ID#:

Email:

Lecture & Tutorial:

| Problem # | Max points possible | Actual score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 10 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 15 | |
| Total | 120 | |

You have **180 minutes** to complete this exam.

**No books, notes, or electronic devices can be used on the test.**

Clearly label your answers by putting them in a $\boxed{\text{box}}$.

Partial credit can be given on some problems if you show your work. Good luck!

**Problem 1.** (3 + 3 + 3 + 3 + 3 = 15 points) Write complete, precise definitions of the following italicised terms.

(1) a *linear transformation* $T$ from a vector space $V$ to a vector space $W$.

A linear transformation $T : V \to W$ is a function with the following properties: (1) $T(u+v) = T(u)+T(v)$ for all $u, v \in V$ and (2) $T(cv) = cT(v)$ for all $c \in \mathbb{R}$ and $v \in V$.

(2) the *span* of a finite set of vectors $v_1, v_2, \ldots, v_n$ in a vector space.

The span of $v_1, v_2, \ldots, v_n$ is the set of all vectors of the form

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

where $c_1, c_2, \ldots, c_n \in \mathbb{R}$.

(3) a *linearly independent* set of vectors $v_1, v_2, \ldots, v_n$ in a vector space.

The vectors $v_1, v_2, \ldots, v_n$ are linearly independent if whenever $c_1, c_2, \ldots, c_n \in \mathbb{R}$ and $c_1v_1 + c_2v_2 + \ldots c_nv_n = 0$, it holds that $c_1 = c_2 = \cdots = c_n = 0$.

(4) a *subspace* $W$ of a vector space $V$.

A subspace $W$ of a vector space $V$ is a subset containing the zero vector in $V$, such that (1) if $u, v \in W$ then $u + v \in W$ and (2) if $c \in \mathbb{R}$ and $v \in W$ then $cv \in W$.

(5) a *basis* for a vector space $V$.

A basis for a vector space $V$ is a linearly independent set of vectors whose span is $V$.

**Problem 2.** (15 points) In the following statements, $A, B, C$, etc., are matrices (with all real entries), and $u, v, w, x,$, etc., are vectors in $\mathbb{R}^n$, unless otherwise noted.

Indicate which of the following is TRUE or FALSE.

One point will be given for each correct answer (no penalty for incorrect answers).

(1) Any system of $n$ linear equations in $n$ variables has at least $n$ solutions.

FALSE (such a system could have 0 solutions)

(2) If a linear system $Ax = b$ has more than one solution, then so does $Ax = 0$.

TRUE (if $Ax = Ay = b$, $x \neq y$, then $A(x-y) = 0$ and $A(0) = 0$)

(3) If $A$ and $B$ are $n \times n$ matrices with $AB = 0$, then $A = 0$ or $B = 0$.

FALSE (take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$)

(4) If $AB = BA$ and $A$ is invertible, then $A^{-1}B = BA^{-1}$.

TRUE ($BA^{-1} = A^{-1}(AB)A^{-1} = A^{-1}(BA)A^{-1} = AB^{-1}$)

(5) If $A$ is a square matrix, then $\det(-A) = -\det A$.

FALSE (if $A$ is $n \times n$, then $\det(-A) = (-1)^n \det A$)

(6) If $A$ is a nonzero matrix then $\det A^T A > 0$.

FALSE

(if $A$ is square then $\det A^T A = (\det A^T)(\det A) = (\det A)^2 \geq 0$; this is zero when $A$ is not invertible)

(7) If $A$ is $m \times n$ and the transformation $x \mapsto Ax$ is onto, then $\operatorname{rank}(A) = m$.

TRUE (onto $\Rightarrow$ $\mathrm{Col}A = \mathbb{R}^m \Rightarrow \operatorname{rank}(A) = \dim \mathrm{Col}A = m$)

(8) If $V$ is a vector space and $S \subset V$ is a subset whose span is $V$, then some subset of $S$ is a basis of $V$.

TRUE (take a minimal subset of $S$ that's linearly indep.)

(9) If $A$ is square and contains a row of zeros, then 0 is an eigenvalue of $A$.

TRUE

($A^T$ has a column of zeros, so $A^T$ is not invertible, so $A^T$ has 0 as an eigenvalue, and $A$ has same eigenvalues as $A^T$)

(10) Each eigenvector of a square matrix $A$ is also an eigenvector of $A^2$.

TRUE (if $Av = \lambda v$ then $A^2 v = A(\lambda v) = \lambda A v = \lambda^2 v$)

(11) If $A$ is diagonalisable, then the columns of $A$ are linearly independent.

FALSE (any zero matrix is diagonal and diagonalisable)

(12) Every $2 \times 2$ matrix (with all real entries) has an eigenvector in $\mathbb{R}^2$.

FALSE

(the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $i$ and $-i$ and no real eigenvectors)

(13) Every $3 \times 3$ matrix (with all real entries) has an eigenvector in $\mathbb{R}^3$.

TRUE

(the characteristic polynomial of such a matrix factors as

$$(\lambda_1 - x)(\lambda_2 - x)(\lambda_3 - x)$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and if $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\}$ then $\overline{\lambda} \in \{\lambda_1, \lambda_2, \lambda_3\}$. Some $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\}$ must therefore have $\lambda = \overline{\lambda} \in \mathbb{R}$, and this real eigenvalue must have a real eigenvector)

(14) If $\|u-v\|^2 = \|u\|^2 + \|v\|^2$ then vectors $u, v \in \mathbb{R}^m$ are orthogonal.

TRUE

(since $\|u-v\|^2 = (u-v) \bullet (u-v) = \|u\|^2 + \|v\|^2 - 2(u \bullet v)$)

(15) If the columns of $A$ are orthonormal then $AA^T$ is an identity matrix.

FALSE

orthonormal columns $\Rightarrow A^T A$ is the identity matrix.
the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ has orthonormal columns and

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I.$$

**Problem 3.** (5 + 5 = 10 points)

(a) Compute the determinant of

$$A = \begin{bmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{bmatrix}$$

where $a, b, c, d$ are real numbers.

For full credit, express your answer in as simple a form as possible.

**Solution:**

$$\begin{aligned} \det A &= -\det \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} \\ &= -\det \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{bmatrix} = \boxed{-(ad-bc)^2} \end{aligned}$$

(b) Find a matrix $M$ such that $M\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $M\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$.

**Solution:**

Such a matrix has $M\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$.

The matrix $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ has $\det A = 16 - 15 = 1$ so is invertible with

$$A^{-1} = \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}.$$

Therefore $\boxed{M = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -11 & 8 \end{bmatrix}}$.

Final step: check that $M\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $M\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$.

**Problem 4.** (5 + 5 + 5 = 15 points) Let $\mathcal{V}$ be the vector space of $3 \times 3$ matrices.

Define $L : \mathcal{V} \to \mathcal{V}$ as the linear transformation $L(A) = A + A^T$.

(a) Find a basis for the subspace $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$. What is $\dim \mathcal{N}$?

**Solution:**

Consider a generic $3 \times 3$ matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. We have

$$L(A) = A + A^T = \begin{bmatrix} 2a & b+d & c+g \\ b+d & 2e & f+h \\ c+g & f+h & 2i \end{bmatrix}.$$

We have $L(A) = 0$ if and only if

$$a = e = i = 0, \qquad b = -d, \qquad c = -g, \quad \text{and} \quad f = -h,$$

i.e., if

$$A = b\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The matrices

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}$$

span $\mathcal{N}$ and are linearly independent, so they form a basis, and $\dim \mathcal{N} = 3$.

(b) Find a basis for the subspace $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$. What is $\dim \mathcal{R}$?

**Solution:**The matrices

$$\boxed{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

span $\mathcal{R}$ and are linearly independent, so they form a basis, and $\dim \mathcal{R} = 6$.

(c) Find two numbers $\lambda, \mu \in \mathbb{R}$ and two nonzero matrices $A, B \in \mathcal{V}$ such that

$$L(A) = \lambda A \quad \text{and} \quad L(B) = \mu B.$$

**Solution:**

We have $L(A) = \lambda A$ for $\lambda = 2$ and $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

We have $L(B) = \mu B$ for $\mu = 0$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

**Problem 5.** (3 + 4 + 4 + 4 = 15 points) Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this problem $A$ refers to a $3 \times 3$ matrix with all real entries satisfying

$$(A - I)(A - 2I)(A - 3I) = 0.$$

(a) Does there exist a $3 \times 3$ matrix $A$ with $(A-I)(A-2I)(A-3I) = 0$ which is not diagonal? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

**Solution:**

The diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has $(D-I)(D-2I)(D-3I) = 0$.

Any similar matrix $A = PDP^{-1}$ has $(A - I)(A - 2I)(A - 3I) = 0$. Take

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{so that} \quad P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and set

$$\boxed{A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}.$$

(b) Does there exist a $3 \times 3$ matrix $A$ with $(A - I)(A - 2I)(A - 3I) = 0$ which has exactly 2 distinct eigenvalues? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

**Solution:**

Take the diagonal matrix $\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$.

(c) Does there exist a $3 \times 3$ matrix $A$ with $(A - I)(A - 2I)(A - 3I) = 0$ which does not have any of the numbers 1, 2, or 3 as an eigenvalue? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

**Solution:**

Suppose $A$ is a $3 \times 3$ with $(A - I)(A - 2I)(A - 3I) = 0$. Then $\det(A - I)\det(A - 2I)\det(A - 3I) = \det((A - I)(A - 2I)(A - 3I)) = \det(0) = 0$ so one of $\det(A - I)$ or $\det(A - 2I)$ or $\det(A - 3I)$ must be zero. Therefore at least one of the numbers 1, 2, and 3 must therefore be an eigenvalue of $A$.

Hence no matrix with the given properties exists.

(d) Does there exist a $3 \times 3$ matrix $A$ with $(A - I)(A - 2I)(A - 3I) = 0$ which is not diagonalisable? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

**Solution:**

Assume $A$ is a 3-by-3 matrix with $(A - I)(A - 2I)(A - 3I) = 0$.

If 1, 2, and 3 are all eigenvalues of $A$ then $A$ is diagonalisable.

Recall that $\lambda$ is not an eigenvalue if and only if $A - \lambda I$ is invertible. If exactly one of the numbers $\lambda \in \{1, 2, 3\}$ is an eigenvalue then $A - \mu I$ would be invertible for the other two numbers $\mu \in \{1, 2, 3\}$, so we could cancel factors in the equation

$$(A - I)(A - 2I)(A - 3I) = 0$$

to deduce that $A - \lambda I = 0$, and hence that $A = \lambda I$ is diagonal and diagonalisable.

The final case to consider is that exactly two numbers $\lambda, \mu \in \{1, 2, 3\}$ are eigenvalues. It would then follow as in the previous paragraph that $(A - \lambda I)(A - \mu I) = 0$. The only way that $A$ could fail to be diagonalisable is if the eigenspaces of $\lambda$ and $\mu$ both have dimension one. In this event, we would have $\dim \mathrm{Nul}(A - \lambda I) = \dim \mathrm{Nul}(A - \mu I) = 1$ and $\dim \mathrm{Col}(A - \lambda I) = \dim \mathrm{Col}(A - \mu I) = 2$ by the rank-nullity theorem. But the only way we can have $(A - \lambda I)(A - \mu I) = 0$ is if $\mathrm{Col}(A - \mu I) \subset \mathrm{Nul}(A - \lambda I)$, which is impossible if $\dim \mathrm{Nul}(A - \lambda I) < \dim \mathrm{Col}(A - \mu I)$.

We conclude that $A$ must be diagonalisable.

**Problem 6.** (4 + 7 + 4 = 15 points)

(a) Compute the distinct eigenvalues of the matrix $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$.

**Solution:**

The characteristic polynomial of $A$ is $(.4 - x)(1.2 - x) + 0.12 = 0.48 - 1.6x + x^2 + 0.12 = 0.60 - 1.6x + x^2 = (1 - x)(0.6 - x)$ so the eigenvalues of $A$ are $\boxed{1 \text{ and } 0.6}$.

(b) Again let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$. Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

**Solution:**

An eigenvector for the eigenvalue 1 of $A$ is a nonzero element of the null space of

$$A - I = \begin{bmatrix} -.6 & -.3 \\ .4 & .2 \end{bmatrix}.$$

The first column is twice the second, so such an eigenvector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

An eigenvector for the eigenvalue 0.6 of $A$ is a nonzero element of the null space of

$$A - .6I = \begin{bmatrix} -.2 & -.3 \\ .4 & .6 \end{bmatrix}.$$

The second column is $1.5$ times the first, so such an eigenvector $\begin{bmatrix} -1.5 \\ 1 \end{bmatrix}$.

One choice for the invertible matrix $P$ and diagonal matrix $D$ is then

$$\boxed{P = \begin{bmatrix} 1 & -1.5 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix}}$$

(c) Continue to let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$.

Find real numbers $a, b, c, d$ such that $\lim_{n\to\infty} A^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

**Solution:**

The inverse of $P$ in the previous part is $P^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & 1.5 \\ 2 & 1 \end{bmatrix}$ and we have

$$A^n = (PDP^{-1})^n = PD^nP^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix}^n P^{-1} = P\begin{bmatrix} 1 & 0 \\ 0 & .6^n \end{bmatrix} P^{-1}.$$

If we take the limit as $n \to \infty$, this becomes

$$\begin{aligned} P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} &= -\frac{1}{2}\begin{bmatrix} 1 & -1.5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1.5 \\ 2 & 1 \end{bmatrix} \\ &= -\frac{1}{2}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1.5 \\ 2 & 1 \end{bmatrix} \\ &= -\frac{1}{2}\begin{bmatrix} 1 & 1.5 \\ -2 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -.5 & -.75 \\ 1 & 1.5 \end{bmatrix}. \end{aligned}$$

So we have $\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -.5 & -.75 \\ 1 & 1.5 \end{bmatrix}.}$

**Problem 7.** (5 + 5 = 10 points)

(a) Find an orthonormal basis for the subspace of vectors of the form

$$\begin{bmatrix} a+2b+3c \\ 2a+3b+4c \\ 3a+4b+5c \\ 4a+5b+6c \end{bmatrix}$$

where $a, b, c$ are real numbers.

**Solution:**

The subspace is the span of $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, $x_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$.

Since $x_2 - x_1 = x_3 - x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, it follows that $x_3 = 2x_2 - x_1$, so the space is spanned by just $x_1$ and $x_2$.

We use the Gram-Schmidt process to covert these vectors to an orthogonal basis $v_1, v_2$.

First, we have $v_1 = x_1$. Second, we have

$$\begin{aligned} v_2 = x_2 - \frac{x_2 \bullet v_1}{v_1 \bullet v_1} v_1 &= \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \frac{2+6+12+20}{1+4+9+16} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \end{aligned}$$

We must normalize $v_1, v_2$ to get an orthonormal basis $u_1, u_2$.

Specifically, we have

$$\boxed{u_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad u_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix}.}$$

(b) Find the vector in $W = \mathbb{R}\text{-span}\{u, v\}$ which is closest to $y$ where

$$y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

**Solution:**

The desired vector is the orthogonal projection of $y$ onto $W$. The vectors $u$ and $v$ are orthogonal, so a formula for this projection is

$$\frac{y \bullet u}{u \bullet u} u + \frac{y \bullet v}{v \bullet v} v = \frac{30}{10} \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} + \frac{26}{26} \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

**Problem 8.** (10 points) Describe all least-squares solutions to the linear equation

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

**Solution:**

The least-squares solutions to $Ax = b$ are the exact solutions to $A^T Ax = A^T b$. We have

$$A^T A = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad A^T b = \begin{bmatrix} 27 \\ 12 \\ 15 \end{bmatrix}.$$

To solve $A^T Ax = A^T b$, we row reduce

$$\left[\begin{array}{ccc|c} 6 & 3 & 3 & 27 \\ 3 & 3 & 0 & 12 \\ 3 & 0 & 3 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 1 & 1 & 0 & 4 \\ 1 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This means that $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a least squares solution if and only if $x_1 + x_3 = 5$ and $x_2 - x_3 = -1$, i.e., when $\boxed{x = \begin{bmatrix} 5-c \\ c-1 \\ c \end{bmatrix} \text{ for any } c \in \mathbb{R}.}$

**Problem 9.** (3 + 5 + 7 = 15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^T A$.

**Solution:**

The matrix $A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is diagonal, so its eigenvalues are 2 and 3.

(b) Find an orthonormal basis $v_1, v_2$ for $\mathbb{R}^2$ consisting of eigenvectors of $A^T A$.

**Solution:**

Since $A^T A$ is diagonal, an orthonormal basis of eigenvectors is

$$\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}.$$

(c) Find a singular value decomposition for $A$. In other words, find the singular values $\sigma_1 \geq \sigma_2$ of $A$ and then express $A$ as a product

$$A = U\Sigma V^T$$

where $U$ and $V$ are invertible matrices with

$$U^{-1} = U^T \quad \text{and} \quad V^{-1} = V^T \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}.$$

**Solution:**

Let $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{3}$ be the singular values of $A$. Then let

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

be the corresponding orthonormal eigenvectors of $A^T A$.

Next define $u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. An orthonormal vector orthogonal to $u_1$ and $u_2$ is

$$u_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The desired matrices $U$, $\Sigma$, and $V$ are then

$$\boxed{U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.}$$