1. Give examples of linear systems in two variables with (a) no solutions, (b) one solutions, (c) infinitely many solutions.

**Solution.** (a) $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1. \end{cases}$ (b) $\begin{cases} x_1 + x_2 = 0 \\ x_1 - x_2 = 1. \end{cases}$ (c) $\begin{cases} x_1 + x_2 = 0 \\ 2x_1 + 2x_2 = 0. \end{cases}$

2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

   (a) Give an example of a linear system whose coefficient matrix is $A$.

   (b) Give an example of a linear system whose augmented matrix is $A$.

   (c) Describe all solutions to the system in (b). How many solutions are there?

**Solution.** (a) $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 3x_2 + 4x_3 = 2 \\ 3x_1 + 4x_2 + 5x_3 = 3. \end{cases}$

(b) $\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + 3x_2 = 4 \\ 3x_1 + 4x_2 = 5. \end{cases}$

(c) We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \texttt{RREF}(A).$$

This shows that columns 1 and 2 are the pivot columns of $A$, so $x_1$ and $x_2$ are both basic variables. Therefore the system in (b) has exactly one solution $(x_1, x_2) = (-1, 2)$.

3. Given the definitions of the following: (a) *row operation*, (b) *echelon form*, (c) *reduced echelon form*, (d) *leading entry*, (e) *pivot position*, (f) *pivot column*, (g) *basic variable*, (h) *free variable*.

**Solution.** See the textbook or lectures notes for definitions.

4. Suppose your phone number is 12345678. Form the $3 \times 3$ matrix

$$A = \begin{bmatrix} x & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

   (You might try this problem with your own phone number instead.)

   (a) Substitute an arbitrary value for $x$, and then compute the reduced echelon form of $A$.

   (b) Find another value for $x$ which results in $A$ having a different reduced echelon form.

   (c) Describe the possible values of $\texttt{RREF}(A)$ as a function of $x$.

**Solution.** (a) Let's try $x = 0$. Then

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Instead let $x = 3$. Then

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 3 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) The second two columns of $A$ are linearly independent since neither is a scalar multiple of the other. If the first column is not in the span of these two columns, then the reduced echelon form of the matrix will be the 3-by-3 identity matrix as in case (b).

So for what values of $x$ is $\begin{bmatrix} x \\ 3 \\ 6 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$? This is the same as asking for the values of $x$ such that the vector equation

$$y_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} x \\ 3 \\ 6 \end{bmatrix} \qquad (*)$$

has a solution. We solve this vector equation by row reduction:

$$\begin{bmatrix} 1 & 2 & x \\ 4 & 5 & 3 \\ 7 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & -3 & 3-4x \\ 0 & -6 & 6-7x \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & 3 & -3+4x \\ 0 & -6 & 6-7x \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & x \\ 0 & 3 & -3+4x \\ 0 & 0 & x \end{bmatrix}.$$

The last matrix is only in echelon form, not reduced echelon form. But from this matrix we can already see that the last column will contain a pivot position precisely when $x \neq 0$. The vector equation (*) has no solution if and only if this happens.

Thus if $x \neq 0$ then $\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and if $x = 0$ then $\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

5. Given the definitions of (a) *linear combination*, (b) *span*, and (c) *linear independence* of a set of vectors.

**Solution.** See the textbook or lectures notes for definitions.

6. Determine if the columns of the matrices

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

are linearly independent.

**Solution.** The columns of a matrix are linearly independent if every column is a pivot column. In this problem

$$\mathtt{RREF}(A) = \mathtt{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so in both matrices the columns are linearly independent.

7. Compute $AB^T$ and $BA^T$, with $A$ and $B$ defined as in the previous problem.

**Solution.** This is just arithmetic. Double check your answer yourself!

8. Do the columns of $A$ or $B$ span $\mathbb{R}^4$?

   Do the columns of $A^T$ or $B^T$ span $\mathbb{R}^3$?

**Solution.** The columns of $A$ do not span $\mathbb{R}^4$ since $A$ does not have a pivot position in every row (only rows 1, 2, and 3). The same is true for $B$.

The columns of both $A^T$ and $B^T$ span $\mathbb{R}^3$. The hard but straightforward way to check this is to compute $\mathtt{RREF}(A^T)$ and $\mathtt{RREF}(B^T)$ and see that there are pivot positions in every row. The easy but slightly tricky way to see this is to note that

$$\mathtt{RREF}(A) = \mathtt{RREF}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

implies that there are matrices 4-by-4 matrix $E$ and $F$ such that

$$EA = FB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so by taking transposes we have

$$A^T E^T = B^T F^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now observe that if $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ is any vector then $A^T x = B^T y = v$ for

$$x = E^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad y = F^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}.$$

9. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function. Say what it means for $f$ to be (a) *linear*, (b) *one-to-one*, (c) *onto*, (d) *invertible*.

**Solution.** See the textbook or lectures notes for definitions.

10. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto and linear. What are the possible values for $n - m$?

**Solution.** If $T$ is onto and linear then $n \geq m$ so the possible values for $n - m$ are $0, 1, 2, 3, 4, 5, \dots$ i.e. any nonnegative integer.

11. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one and linear. What are the possible values for $n - m$?

**Solution.** If $T$ is one-to-one and linear then $n \leq m$ so the possible values for $n-m$ are $0, -1, -2, -3, -4, -5, \dots$ i.e. any nonpositive integer.

12. Determine if the matrix

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \end{bmatrix}$$

is invertible. If it is, compute its inverse.

**Solution.** We can check if $A$ is invertible and compute its inverse at the same time by row reducing

$$\begin{bmatrix} 0 & -8 & 5 & 1 & 0 & 0 \\ 3 & -7 & 4 & 0 & 1 & 0 \\ -1 & 5 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 7/24 & -1/8 \\ 0 & 1 & 0 & -1/3 & -5/24 & -5/8 \\ 0 & 0 & 1 & -1/3 & -1/3 & -1 \end{bmatrix}.$$

(I'm not showing my work here, but you should!) Since the first three columns give the identity matrix, $\texttt{RREF}(A) = I_3$ so $A$ is invertible, with inverse

$$A^{-1} = \frac{1}{24} \begin{bmatrix} -8 & 7 & -3 \\ -8 & -5 & -15 \\ -8 & -8 & -24 \end{bmatrix}.$$

13. Consider the matrix

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & h & i \end{bmatrix}.$$

What is $\det A$? When is $A$ invertible? Assuming $A$ invertible, given a formula for $A^{-1}$.

**Solution.** Observe that $A = BCD$ where

$$B = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & h & i \end{bmatrix}.$$

The determinant of each of these is easy compute using the recursive rule for determinants:

$$\det B = ad - bc \qquad \text{and} \qquad \det C = e \qquad \text{and} \qquad \det d = fi - gh.$$

Therefore $\det A = (ad - bc)(e)(fi - gh)$.

The matrix $A$ is invertible if $ad - bc$ and $e$ and $fi - gh$ are all nonzero. In this case

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 & 0 & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 & 0 \\ 0 & 0 & 1/e & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{fi-gh} & \frac{-g}{fi-gh} \\ 0 & 0 & 0 & \frac{-h}{fi-gh} & \frac{f}{fi-gh} \end{bmatrix}.$$

14. Given the definition of the following (a) *subspace* of $\mathbb{R}^n$, (b) *basis* of a subspace, (c) *dimension* of a subspace.

**Solution.** See the textbook or lectures notes for definitions.

15. Let $A$ be an $m \times n$ matrix. Given the definition of the following (a) the *nullspace* of $A$, (b) the *column space* of $A$, and (c) the *rank* of $A$.

**Solution.** See the textbook or lectures notes for definitions.

16. Suppose $A$ is an $m \times n$ matrix. What are the possible values for $\operatorname{rank} A$? What are the possible values of $\dim \operatorname{Nul} A$?

**Solution.** $\operatorname{rank} A$ can be $0, 1, 2, 3, \dots,$ or $m$. $\dim \operatorname{Nul} A$ can be $0, 1, 2, 3, \dots,$ or $n$.

17. Find a basis for the nullspace of

$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}.$$

**Solution.** This was not a very interesting matrix to consider. It follows from Problem 12 that

$$\texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the columns of $A$ are linearly independent and $\operatorname{Nul} A = \{0\}$, so the empty set is a basis for $\operatorname{Nul} A$.

18. Find a basis for the column space of $A^T$, with $A$ defined as in the previous problem.

**Solution.** We have

$$\texttt{RREF}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & -1/4 \end{bmatrix}$$

so the first three columns $\begin{bmatrix} 0 \\ -8 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$ are a basis for $\operatorname{Col} A^T = \mathbb{R}^3$.

A more interesting question would be to ask for a basis of $\operatorname{Nul} A^T$. By the rank theorem, $\operatorname{Nul} A^T$ is 1-dimensional since $\dim \operatorname{Col} A^T + \dim \operatorname{Nul} A^T = 4$. So a basis of $\operatorname{Nul} A$ is given by any nonzero element in the subspace. For example, the vector $\begin{bmatrix} 0 \\ -1 \\ 1 \\ 4 \end{bmatrix}$.

19. Is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 3 \\ 4 & v_2 & 6 \\ 7 & v_3 & 9 \end{bmatrix}$$

a linear transformation $\mathbb{R}^3 \to \mathbb{R}$? If it is, compute its standard matrix.

**Solution.** Yes it is; this is one of the defining properties of the determinant.

We have

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 3 \\ 4 & v_2 & 6 \\ 7 & v_3 & 9 \end{bmatrix} = (9v_2-6v_3)-v_1(36-42)+3(4v_3-7v_2) = 6v_1-12v_2+6v_3$$

so the standard matrix of $T$ is $A = \begin{bmatrix} 6 & -12 & 6 \end{bmatrix}$.

20. Suppose you have two matrices $A$ and $B$ of the same size. How would you construct a matrix $C$ whose nullspace is the intersection of $\operatorname{Nul} A$ and $\operatorname{Nul} B$?

**Solution.** Stack the two matrices on top of each other to form $C = \begin{bmatrix} A \\ B \end{bmatrix}$. Then $Cv = \begin{bmatrix} Av \\ Bv \end{bmatrix} = 0$ if and only if $Av = 0$ and $Bv = 0$, so $v \in \operatorname{Nul} C$ if and only if $v \in \operatorname{Nul} A$ and $v \in \operatorname{Nul} B$, so $\operatorname{Nul} C = \operatorname{Nul} A \cap \operatorname{Nul} B$.

21. Suppose you have two matrices $A$ and $B$ of the same size. How would you construct a matrix $C$ whose column space contains both $\operatorname{Col} A$ and $\operatorname{Col} B$?

Just put the two matrices side by side to form $C = \begin{bmatrix} A & B \end{bmatrix}$. The columns of $C$ the include all columns of $A$ and $B$, and therefore $\operatorname{Col} C$ contains all linear combinations of these columns.

22. Compute the determinant of

$$A = \begin{bmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}.$$

**Solution.** Use the recursive determinant formula to get $\det A = -xyzw$. Alternatively,

$$\det A = xyzw \det \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -xyzw$$

since the permutation matrix has inversion number 1.

23. Does there exist a $2 \times 2$ matrix $A$ with all entries in $\mathbb{R}$ such that $A^2 v = -v$ for all $v \in \mathbb{R}^2$? If not, say why. If there is, give an example. (Recall that $A^2 = AA$ for a square matrix.)

**Solution.** If $A^2v = -v$ for $v \in \mathbb{R}^2$ then multiplication by $A^2$ acts on the $\mathbb{R}^2$-plane by rotating everything 180 degrees counterclockwise: this reverses the direction of all vectors, sending $v$ to $-v$.

The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ acts on $\mathbb{R}^2$ by rotating all vectors counterclockwise by 90 degrees. Therefore $A^2$ acts to rotate a given vector 90 degrees twice, i.e., 180 degrees. You can also check directly that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It actually follows from this exercise (with some extra work) that for any $k > 1$, there is a $k \times k$ matrix $X$ with all entries in $\mathbb{R}$ satisfying any polynomial equation of the form

$$a_n X^n + a_{n-1} X^{n-1} + \cdots + a_2 X^2 + a_1 X + a_0 I_k = 0$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. This is not true if $k = 1$, since for example $X^2 + 1 = 0$ has no real solutions $X \in \mathbb{R}$.