**Note:** This long list of problems covers all of the material you should review for the midterm. These problems are similar in nature to ones that will appear on the exam, though the test will not have anywhere near this many questions.

1. Give examples of linear systems in two variables with (a) no solutions, (b) one solutions, (c) infinitely many solutions.

2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$.

   (a) Give an example of a linear system whose coefficient matrix is $A$.

   (b) Give an example of a linear system whose augmented matrix is $A$.

   (c) Describe all solution to the system in (b). How many solutions are there?

3. Given the definitions of the following: (a) *row operation*, (b) *echelon form*, (c) *reduced echelon form*, (d) *leading entry*, (e) *pivot position*, (f) *pivot column*, (g) *basic variable*, (h) *free variable*.

4. Suppose your phone number is 12345678. Form the $3 \times 3$ matrix
$$A = \begin{bmatrix} x & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

   (You might try this problem with your own phone number instead.)

   (a) Substitute an arbitrary value for $x$, and then compute the reduced echelon form of $A$.

   (b) Find another value for $x$ which results in $A$ having a different reduced echelon form.

   (c) Describe the possible values of $\texttt{RREF}(A)$ as a function of $x$.

5. Given the definitions of (a) *linear combination*, (b) *span*, and (c) *linear independence* of a set of vectors.

6. Determine if the columns of the matrices
$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$
   are linearly independent.

7. Compute $AB^T$ and $BA^T$, with $A$ and $B$ defined as in the previous problem.

8. Do the columns of $A$ or $B$ span $\mathbb{R}^4$?

   Do the columns of $A^T$ or $B^T$ span $\mathbb{R}^3$?

9. Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function. Say what it means for $f$ to be (a) *linear*, (b) *one-to-one*, (c) *onto*, (d) *invertible*.

10. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto and linear. What are the possible values for $n - m$?

11. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one and linear. What are the possible values for $n - m$?

12. Determine if the matrix
$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \end{bmatrix}$$
is invertible. If it is, compute its inverse.

13. Consider the martrix
$$A = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & h & i \end{bmatrix}.$$
When is $A$ invertible? Assuming $A$ invertible, given a formula for $A^{-1}$. What is $\det A$?

14. Give the definition of the following (a) *subspace* of $\mathbb{R}^n$, (b) *basis* of a subspace, (c) *dimension* of a subspace.

15. Let $A$ be an $m \times n$ matrix. Given the definition of the following (a) the *nullspace* of $A$, (b) the *column space* of $A$, and (c) the *rank* of $A$.

16. Suppose $A$ is an $m \times n$ matrix. What are the possible values for $\operatorname{rank} A$? What are the possible values of $\dim \operatorname{Nul} A$?

17. Find a basis for the nullspace of
$$A = \begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}.$$

18. Find a basis for the column space of $A^T$, with $A$ defined as in the previous problem.

19. Is the function
$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 3 \\ 4 & v_2 & 6 \\ 7 & v_3 & 9 \end{bmatrix}$$
a linear transformation $\mathbb{R}^3 \to \mathbb{R}$? If it is, compute its standard matrix.

20. Suppose you have two matrices $A$ and $B$ of the same size. How would you construct a matrix $C$ whose nullspace is the intersection of $\operatorname{Nul} A$ and $\operatorname{Nul} B$?

21. Suppose you have two matrices $A$ and $B$ of the same size. How would you construct a matrix $C$ whose column space contains both $\operatorname{Col} A$ and $\operatorname{Col} B$?

22. Compute the determinant of
$$A = \begin{bmatrix} 0 & x & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix}.$$

23. Does there exist a $2 \times 2$ matrix $A$ with all entries in $\mathbb{R}$ such that $A^2 v = -v$ for all $v \in \mathbb{R}^2$? If not, say why. If there is, give an example. (Recall that $A^2 = AA$ for a square matrix.)