1 Last time: row reduction to (reduced) echelon form

The *leading entry* in a nonzero row of a matrix is the first nonzero entry from left going right. For example, the row $\begin{bmatrix} 0 & 0 & 7 & 0 & 5 \end{bmatrix}$ has leading entry 7, which occurs in the 3rd column.

Definition. A matrix with m rows and n columns is in *echelon form* if it has the following properties:

1. If a row is nonzero, then every row above it is also nonzero.
2. The leading entry in a nonzero row is in a column to the right of the leading entry in the row above.
3. If a row is nonzero, then every entry below its leading entry in the same column is zero.

For example,

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (*)$$

is in echelon form, but none of

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 0 & 0 & 4 & 5 \end{bmatrix}$$

is in echelon form.

Definition. A matrix is in *reduced echelon form* if

1. The matrix is in echelon form.
2. Each nonzero row has leading entry 1.
3. The leading 1 in each nonzero row is the only nonzero number in its column.

The matrix

$$\begin{bmatrix} 1 & 0 & -10/3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in reduced echelon form and is row equivalent to the matrix (*).

Theorem. Each matrix A is row equivalent to exactly one matrix in reduced echelon form.

We denote this matrix by $\texttt{RREF}(A)$.

The *row reduction algorithm* is a way of constructing $\texttt{RREF}(A)$ from A. This algorithm is something you should memorize and be able to perform quickly. We won't review the full definition again in this lecture, but let's do an example.

Example. Writing $\to$ to indicate a sequence of row operations, we have

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 3 & 9 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and the last matrix is the reduced echelon form of the first matrix.

Remark. There is another way to think about what this computation means.

Note that the first matrix is the augmented matrix of the linear system

$$\begin{aligned} a + b + c &= 0 \\ a + 2b + 2^2c &= 1 \\ a + 3b + 3^2c &= 2 \end{aligned}$$

where we are now using a, b, c as variables rather than x_1, x_2, x_3 as usual.

A solution to this linear system gives the coefficients of a polynomial $f(x) = a + bx + cx^2$ with $f(1) = 0$, $f(2) = 1$, and $f(3) = 2$. The graph of this function is a parabola passing through the points $(1, 0)$, $(2, 1)$, and $(2, 2)$. But these three points are all on the same line $y = x - 1$.

We therefore must have $f(x) = x - 1$ and $(a, b, c) = (-1, 1, 0)$ must be the unique solution to our system. This forces the reduced echelon form of our augmented matrix to be what we computed.

A *pivot column* of a matrix A is a column containing a leading 1 in $\texttt{RREF}(A)$.

If A is the augmented matrix of a linear system in variables $x_1, x_2, \dots, x_n$, then we say that x_i is a *basic variable* if i is a pivot column and that x_i is a *free variable* if i is not a pivot column.

To determine the basic and free variables of the system, we have to perform the row reduction algorithm to figure out what $\texttt{RREF}(A)$ is first. Once we have done this, we can conclude that:

- The system has 0 solutions if the last column is a pivot column of A.
- The system has ∞ solutions if the last column is not a pivot column but there is ≥ 1 free variable.
- The system has 1 solution if there are no free variables, and the last column is not a pivot column.

Moreover, here's how you find all the solutions to the system: choose any values for the free variables, then solve for the basic variables in terms of the free variables via the equations which make up the linear system corresponding to $\texttt{RREF}(A)$.

2 Vectors

Until we see vector spaces later in this course, the term *vector* will always refer to an ordered list of numbers in $\mathbb{R}$. A *vector* (sometimes to be called a *column vector*) is such a list oriented vertically; in other words, a matrix with one column:

$$\begin{bmatrix} 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sqrt{7} \\ \sqrt{6} \end{bmatrix}.$$

We write a general column vector as

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where each v_i is a real number. Two vectors u and v are equal if they have the same number of rows and the same entries in each row.

The sum of two vectors is

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}.$$

Note: $u + v = v + u$, but we can only add together vectors with the same number of rows.

If v is a vector and $c \in \mathbb{R}$ is a *scalar*, i.e., a real number, then we define

$$cv = c\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

We call the new vector cv the *scalar multiple* of v by c.

Example. We have

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

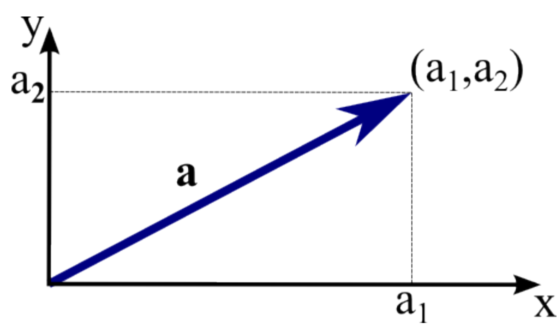
and

$$-\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

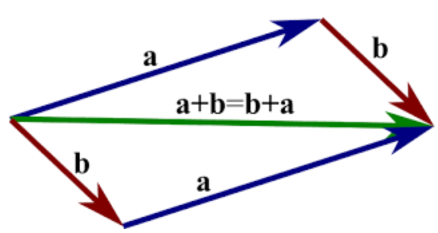
Define *subtraction* of vectors as addition after multiplying by the scalar -1:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}.$$

We write $\mathbb{R}^n$ for the set of all vectors with exactly n rows. Vectors $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$ can be identified with arrows in the Cartesian plane from the origin to the point $(x, y) = (a_1, a_2)$:



Proposition. The sum $a + b$ of two vectors $a, b \in \mathbb{R}^2$ is the vector represented by the arrow from the origin to the point which is the opposite vertex of the parallelogram with sides a and b:



Proof. We have $\frac{a_2}{a_1} = \frac{(a_2+b_2)-b_2}{(a_1+b_1)-b_1}$ and $\frac{b_2}{b_1} = \frac{(a_2+b_2)-a_2}{(a_1+b_1)-a_1}$.

The fractions $\frac{a_2}{a_1}$ and $\frac{b_2}{b_1}$ are the slopes of the lines through the origin containing the vectors a and b.

The other two fractions are the slopes of the lines (1) between the endpoints of b and $a + b$ and (2) between the endpoints of a and $a + b$.

The first line of the proof shows that line (1) is parallel to a, and line (2) is parallel to b.

Therefore lines (1) and (2) are the other two sides of the unique parallelogram with sides a and b.

The endpoint of $a+b$ is where lines (1) and (2) intersect.

Therefore this endpoint is the vertex of the parallelogram opposite the origin. $\square$

The *zero vector* $0 \in \mathbb{R}^n$ is the vector

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

whose entries are all zero. We have $0 + v = v + 0 = v$ for any vector v.

Definition. Suppose $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are vectors and $c_1, c_2, \dots, c_p \in \mathbb{R}$ are *scalars*, i.e., numbers. The vector $y = c_1v_1 + c_2v_2 + \dots + c_pv_p$ is called a *linear combination* of $v_1, v_2, \dots, v_p$. It is the linear combination of $v_1, v_2, \dots, v_p$ with *coefficients* $c_1, c_2, \dots, c_p$.

Example. Suppose $a = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $c = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Is c a linear combination of a and b?

If it were, we could find numbers $x_1, x_2 \in \mathbb{R}$ such that $x_1a + x_2b = c$, i.e., such that

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3. \end{aligned}$$

So to answer our question we need to determine if this linear system has a solution.

To do this, use row reduction:

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \texttt{RREF}(A) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns of A are 1 and 2: the last column is *not* a pivot column. Therefore our linear system is consistent, which means that c *is* a linear combination of a and b.

We generalize this example with the following statement.

Proposition. A vector equation of the form $x_1a_1 + x_2a_2 + \dots + x_na_n = b$ where $x_1, x_2, \dots, x_n$ are variables and $a_1, a_2, \dots, a_n, b \in \mathbb{R}^m$ are vectors, has the same solutions as those for the linear system with augmented matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n & b \end{bmatrix}. \tag{*}$$

This notation means the matrix whose ith column is a_i and last column is b.

In other words, the vector b is a linear combination of $a_1, a_2, \dots, a_n$ if and only if the linear system whose augmented matrix is (*) is consistent.

Definition. The *span* of a vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ is the set of all vectors $y \in \mathbb{R}^n$ that are linear combinations of $v_1, v_2, \dots, v_p$. We denote the span of some set of vectors by

$$\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\} \qquad \text{or} \qquad \text{span}\{v_1, v_2, \dots, v_p\}.$$

What does $\mathbb{R}$-span$\{v_1, v_2, \dots, v_p\}$ look like?

We can visualize the span of the 0 vector as the single point consisting of just the origin. We imagine the span of a collection of vectors that all belong to the same line through the origin as that line.

In $\mathbb{R}^2$, if the span of $v_1, v_2, \dots, v_p$ does not consist of a line, then the span is the whole plane.

To see this, imagine we have two vectors $u, v \in \mathbb{R}^2$ which are not parallel. We can then get to any point in the plane by travelling some distance in the u direction, then some distance in the v direction. In other words, we can get any vector in $\mathbb{R}^2$ as the linear combination $au + bv$ for some scalars $a, b \in \mathbb{R}$. Draw a picture to illustrate this to yourself:

3 Vocabulary

Keywords from today's lecture:

1. **Vector**.

   A vertical list of numbers. Equivalently, a matrix with one column.

   The set of all vectors with n rows is written $\mathbb{R}^n$.

   Example: $\begin{bmatrix} 1 \\ 0 \\ -5.2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 4 \end{bmatrix}$ or $\begin{bmatrix} \sqrt{2} \\ \pi \end{bmatrix}$.

2. **Scalar**.

   Another word for "number" or "constant." We can multiply scalars together, but not vectors.

   Example: 5 or π or $\sqrt{2}$.

3. The **zero vector** $0 \in \mathbb{R}^n$.

   The vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ with n rows all equal to zero.

4. **Linear combination** of vectors.

   If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are vectors, then $u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.

   If $c \in \mathbb{R}$ is a scalar then $cv = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$.

   The linear combination of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ with coefficients $a_1, a_2, \dots, a_p \in \mathbb{R}$ is

   $$a_1 v_1 + a_2 v_2 + \cdots + a_p v_p \in \mathbb{R}^n.$$

   Example: $2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \pi \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 - 0 + \pi \\ 8 - 1 + 3\pi \end{bmatrix} = \begin{bmatrix} 2 + \pi \\ 7 + 3\pi \end{bmatrix}$.

5. The **span** of a list of vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$.

   The set of all linear combinations of the vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$.

   A vector $u \in \mathbb{R}^n$ belongs to the span of $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ if and only if the $n \times (p+1)$ matrix

   $$A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p & u \end{bmatrix}$$

   is the augmented matrix of a consistent linear system.

   This happens precisely when A has no pivot positions in the last column.