1 Last time: multiplying vectors matrices

Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ we define

$$\boxed{Av = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.}$$

We refer to Av as the product of A and v, or the vector given by multiplying v by A.

Example. We have $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ -5+0+7 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

If A is an $m \times n$ matrix and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $b \in \mathbb{R}^m$, then we call $\boxed{Ax = b}$ a *matrix equation.*

A matrix equation $Ax = b$ has the same solutions as the linear system with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$.

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
2. The span of the columns of A is all of $\mathbb{R}^m$.
3. A has a pivot position in every row.

Example. The matrix equation

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

may fail to have a solution since

$$\texttt{RREF}\left(\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$$

has pivot positions only in rows 1 and 2.

A *homogeneous* linear system is one that can be written $Ax = 0$.

Such a system has one *trivial solution* given by $x = 0$.

A homogeneous linear system has a nontrivial solution if and only if it has at least one free variable.

A homogeneous linear system has a free variable if not every column is a pivot column in its *coefficient matrix*, which is the augmented matrix without the last column.

2 Linear independence

We briefly introduced the notion of linear independence last time.

Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are *linearly independent* if the homogeneous matrix equation

$$\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0$$

has no nontrivial solution.

If $c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$ where $c_1, c_2, \dots, c_p \in \mathbb{R}$ and some $c_i \neq 0$, then we refer to "$c_1v_1 + c_2v_2 + \dots + c_pv_p = 0$" as a *linear dependence* among the vectors $v_1, v_2, \dots, v_p$.

Vectors are *linearly independent* if there is no linear dependence among them.

Vectors which are not linearly independent are *linearly dependent.*

How to determine if $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linear independent.

- Form the $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$.
- Reduce A to echelon form (or to reduced echelon form) to find its pivot columns.
- If every column of A is a pivot column, then the vectors are linearly independent.

  If some column of A is not a pivot column, then the vectors are linearly dependent.

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$ are linear dependent since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \texttt{RREF}(A)$$

where $\sim$ denotes row equivalence. The last matrix has no pivot position in column 3. In fact, we have

$$-\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 15 \end{bmatrix}$ are linearly independent, since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \texttt{RREF}(A)$$

Every column of A contains a pivot position, so the linear system with coefficient matrix A has no free variables, so $Ax = 0$ have no nontrivial solutions, meaning the columns of A are linearly independent.

Facts about linear independence.

1. A single vector v is linearly independent if and only if $v \neq 0$.

   A list of vectors is linearly dependent if it includes the 0 vector.

2. Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are linearly dependent if and only if some vector v_i is a linear combination of the other vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p$.

   We saw this in the previous example: $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

The last thing we'll note about linear independence (for now) is this useful, non-obvious fact:

Theorem. Assume $p > n$ and $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. Then these vectors are linearly dependent.

Proof. Let $A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$.

This matrix has more columns than rows.

Each row contains at most one pivot position, so there are fewer pivot positions than columns.

Therefore some column is not a pivot column.

This means the linear system $Ax = 0$ has a free variable, so has a nontrivial solution.

This implies that $v_1, v_2, \dots, v_p$, the columns of A, are linearly dependent. $\square$

Example. The vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 5 \\ 60 \end{bmatrix}$ are linearly dependent since $3 > 2$.

3 Linear transformations

A function f (like the ones we see in calculus) takes an input x from some set X (for example, $\mathbb{R}$) and produces an output $f(x)$ in another set Y

We write $f : X \to Y$ to mean that f is a function that takes inputs from X and gives outputs in Y.

X is called the *domain* of the function f.

Y is sometimes called the *codomain* of f.

For every x in the domain X of f, we get an output $f(x)$.

It is possible that some values y in the codomain Y may never occur as outputs of f, however.

The *image* of an input x in X under f is the ouput $f(x)$.

Define the *image* or *range* of the function f to be the subset $\{f(x) : x \in X\}$ of the codomain Y. This is the set of all possible outputs of f. We denote the range of f by $\mathrm{range}(f)$.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose domain and codomain are sets of vectors. The function f is a *linear transformation* or a *linear function* if both of these properties hold:

(1) $f(u + v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(2) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

Example. If A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the function with $T(v) = Av$ for $v \in \mathbb{R}^n$, then T is a linear function.

Linear transformations have the following additional properties:

Proposition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

(3) $f(0) = 0$.

(4) $f(u - v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

(5) $f(au + bv) = af(u) + bf(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

Proof. (3) We have $f(0) = f(0 + 0) = 2f(0)$ so $f(0) = 0$.

(4) We have $f(u - v) = f(u) + f(-v) = f(u) + (-1)f(v) = f(u) - f(v)$.

(5) We have $f(au + bv) = f(au) + f(bv) = af(u) + bf(v)$. □

Example. Suppose $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function defined by $T(v) = Av$.

(a) The image of a vector $v \in \mathbb{R}^2$ under T is by definition $T(v) = Av$.

The image of $v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ under T is $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$.

(b) Is the range of T all of $\mathbb{R}^3$? If it was, then (from results last time) A would have to have a pivot position in every row. This is impossible since each column can only contain one pivot position, but A has three rows and only two columns. Therefore range$(T) \neq \mathbb{R}^3$.

The fundamental theorem relating matrices and linear transformations:

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

<u>Moral:</u> Matrices uniquely represent all linear transformations $\mathbb{R}^n \to \mathbb{R}^m$.

Proof. Define $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ as the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so that e_i has a 1 in the ith row and 0 in all other rows.

Define $a_i = T(e_i) \in \mathbb{R}^m$ and $A = \begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$.

If w is any vector $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{R}^n$ then

$$\begin{aligned} T(w) &= T(w_1e_1 + w_2e_2 + \cdots + w_ne_n) \\ &= w_1T(e_1) + w_2T(e_2) + \cdots + w_nT(e_n) = w_1a_1 + w_2a_2 + \cdots + w_na_n = Aw. \end{aligned}$$

Thus A is one matrix such that $T(v) = Av$ for all vectors $v \in \mathbb{R}^n$.

To show that A is the only such matrix, suppose B is a $m \times n$ matrix with $T(v) = Bv$ for all $v \in \mathbb{R}^n$.

Then $T(e_i) = Ae_i = Be_i$ for all $i = 1, 2, \dots, n$.

But Ae_i and Be_i are the ith columns of A and B. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} e_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Therefore A and B have the same columns, so they are the same matrix: $A = B$. $\square$

We call the matrix A in this theorem the *standard matrix* of the linear transformation T.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $T(v) = 3v$.

This is a linear transformation. (Why?) What is the standard matrix A of T?

As we saw in the proof of the theorem, the standard matrix of $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix} = \begin{bmatrix} 3e_1 & 3e_2 & \dots & 3e_n \end{bmatrix} = \begin{bmatrix} 3 & 0 & \dots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}.$$

In words, A is the matrix with 3 in each position $(1,1), (2,2), \dots, (n,n)$ and 0 in all other positions.

One calls such a matrix *diagonal*.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \cdots + v_n^2.$$

This function is *not* linear: we have $T(2v) = 4T(v) \neq 2T(v)$ for any nonzero vector $v \in \mathbb{R}^n$.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}.$$

This function is a linear transformation. (Why?) Its standard matrix is

$$A = \left[\begin{array}{ccccc} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{array}\right] = \left[\begin{array}{ccccc} e_n & e_{n-1} & \dots & e_2 & e_1 \end{array}\right] = \left[\begin{array}{ccccc} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{array}\right].$$

In the matrix on the right, we adopt the convention of only writing the nonzero entries: all positions in the matrix which are blank contain zero entries.

Example. Fix $\theta \in [0, 2\pi)$. The notation $[a, b)$ means "the set of numbers $x \in \mathbb{R}$ with $a \leq x < b$." Define

$$A = \left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$$

and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(v) = Av$.

If $v = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$ is a vector parallel to the x-axis, then $T(v) = Av = \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right]$.

If $v = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$ is a vector parallel to the y-axis, then $T(v) = Av = \left[\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right] = \left[\begin{array}{c} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{array}\right]$.

In general, $T(v) = Av$ is the vector obtained by rotating v counterclockwise by the angle θ.

This holds since any vector $v = \left[\begin{array}{c} v_1 \\ v_2 \end{array}\right]$ can be written $v = \left[\begin{array}{c} v_1 \\ 0 \end{array}\right] + \left[\begin{array}{c} 0 \\ v_2 \end{array}\right]$, so is the arrow to the opposite vertex in the parallelogram with sides $\left[\begin{array}{c} v_1 \\ 0 \end{array}\right]$ and $\left[\begin{array}{c} 0 \\ v_2 \end{array}\right]$. Since $T(v) = T\left(\left[\begin{array}{c} v_1 \\ 0 \end{array}\right]\right) + T\left(\left[\begin{array}{c} 0 \\ v_2 \end{array}\right]\right)$ and since T rotates by angle θ the two vectors on the right, it follows that $T(v)$ is the arrow from 0 to the opposite vertex in our previous parallelogram, now rotated counterclockwise by angle θ.

4 One-to-one and onto functions

This section talks about two important classes of linear transformations, which can be characterized in terms of whether the columns of the standard matrix are linearly independent or span the codomain.

Definition. A function $f : X \to Y$ is *one-to-one* or *injective* if $f(a) = f(b)$ implies $a = b$. In words: f does not send two different inputs to the same output. If $a \neq b$ and $f(a) = f(b)$ then f is *not one-to-one.*

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \left[\begin{array}{ccc} 1 & 2 & 5 \\ 0 & 5 & 3 \end{array}\right].$$

Is T one-to-one? No: since A has more columns than rows, its columns are linearly dependent. Therefore there is a vector $0 \neq v \in \mathbb{R}^3$ such that $T(v) = Av = 0$. But we also have $T(0) = 0$

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then T is one-to-one if and only if the only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$, i.e., the columns of the standard matrix A of T are linearly independent.

Proof. Suppose the only solution to $T(x) = 0$ is $x = 0 \in \mathbb{R}^n$. Then whenever $u, v \in \mathbb{R}^n$ are vectors with $u \neq v$, we have $T(u) - T(v) = T(u - v) \neq 0$ since $u - v \neq 0$, so $T(u) \neq T(v)$. Therefore T is one-to-one.

If T is one-to-one, then $T(x) = T(0) = 0$ implies $x = 0$, so $T(x) = 0$ has only trivial solutions. $\square$

Definition. A function $f : X \to Y$ is *onto* or *surjective* if $\text{range}(f) = \{f(x) : x \in X\} = Y$. In words: the range of f is equal to its codomain. If there is a value $y \in Y$ such that $f(x) \neq y$ for all $x \in X$, then f is *not onto*.

Example. Suppose again that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is the linear transformation $T(v) = Av$ where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix}.$$

Is T onto? Yes: the columns of A span $\mathbb{R}^2$ if and only if A has a pivot position in every row, and we have

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3/5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 19/5 \\ 0 & 1 & 3/5 \end{bmatrix} = \texttt{RREF}(A).$$

Theorem. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then T is onto if and only if the columns of the standard matrix A of T span $\mathbb{R}^m$.

Proof. The vectors in the range of T are precisely the linear combinations of the columns of A.

The range is $\mathbb{R}^m$ precisely when the span of the columns of A is $\mathbb{R}^m$. □

Example. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 + v_2 \\ 5v_1 + 7v_2 \\ v_1 + 3v_2 \end{bmatrix}.$$

This function is a linear transformation. Its standard matrix is

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}.$$

To determine if T is one-to-one, we check if the columns of A linearly independent. To do this, we row reduce to echelon form:

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & -8 \\ 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \texttt{RREF}(A).$$

This shows that A has a pivot position in every column, which means $Ax = 0$ has only trivial solutions, which means the columns of A are linearly independent, which means T is one-to-one.

To determine if T is onto, we want to find out if the columns of A span $\mathbb{R}^3$. From last time, we know that this happens if and only if A has a pivot position in every row. Since the third row of A has no pivot position, T is not onto.

Corollary. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one only if $n \leq m$, and onto only if $n \geq m$.

Proof. Results last time show that T is one-to-one iff its standard matrix has a pivot position in every column, and onto iff its standard matrix has a pivot position in every row. The first case requires there to be more columns n than rows m, and the second case requires there to be more rows m than columns n (since each row and each column contains at most one pivot position). □

5 Geometric interpretations of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix A. We can illustrate T by drawing the parallelogram with sides $T(e_1)$ and $T(e_2)$. (Fill in these pictures yourself.)

Standard matrix of T	Picture	Description of T
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$		Reflect across the x-axis
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$		Reflect across y-axis
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		Reflect across $y = x$
$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ $(0 < k < 1)$		Horizontal contraction
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ $(0 < k < 1)$		Vertical contraction
$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ $(k > 0)$		Horizontal sheering

6 Vocabulary

Keywords from today's lecture:

1. **Linearly independent** vectors.

   Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are **linearly independent** if $x_1 v_1 + \dots + x_p v_p = 0$ holds only if $x_1 = x_2 = \dots = x_p = 0$; or when $\begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has a pivot position in every column.

   Vectors that are not linearly independent are **linearly dependent**.

   Example: The three vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent.

   The four vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$ are linearly dependent.

2. **Domain** and **codomain** of a function $f : X \to Y$.

   The **domain** X is the set of inputs for the function.

   The **codomain** Y is a set that contains the output of the function. This set can also contain elements that are not outputs of the function.

   Example: If A is an $m \times n$ matrix then the function $T(v) = Av$ has domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$.

3. **Range** of a function $f : X \to Y$.

   The set $\mathrm{range}(f) = \{f(x) : x \in X\} \subset Y$ of all possible outputs of the function f.

   Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $T(v) = Av$ then $\mathrm{range}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$.

4. **Linear function** $f : \mathbb{R}^n \to \mathbb{R}^m$.

   A function with $f(cv) = cf(v)$ and $f(u + v) = f(u) + f(v)$ for $c \in \mathbb{R}$ and $u, v \in \mathbb{R}^n$.

   Example: Every such function has the form $f(v) = Av$ for a unique $m \times n$ matrix A.

   The matrix A is called the **standard matrix** of f if $f(v) = Av$ for all $v \in \mathbb{R}^n$.

5. **Diagonal** matrix

   A matrix which has 0 in position (i, j) if $i \neq j$.

   Example: $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

6. **One-to-one** or **injective** function $f : X \to Y$.

   A function with the property that if $f(u) = f(v)$ for $u, v \in X$ then $u = v$.

   Example: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.
   The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one: $f(-2) = f(2) = 4$.

7. **Onto** or **surjective** function $f : X \to Y$.

   A function with the property that $y \in Y$ then there exists $x \in X$ with $f(x) = y$.

   Example: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.
   The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not onto: no negative number is in its range.