1 Last time: introduction to determinants

Let n be a positive integer.

A *permutation matrix* is a square matrix whose entries are all 0 or 1, and that has exactly one nonzero entry in each row and in each column. Let S_n be the set of $n \times n$ permutation matrices.

If A is an $n \times n$ matrix and $X \in S_n$, then AX has the same columns as A but in a different order: the columns of A are "permuted" by X.

Example. The six elements of S_3 are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Given $X \in S_n$ and an arbitrary $n \times n$ matrix A:

- Define $\Pi(X, A)$ as the product of the entries of A in the nonzero positions of X.
- Define $\mathrm{inv}(X)$ as the number of 2×2 submatrices of X equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

To form a 2×2 submatrix of X, choose any two rows and any two columns, not necessarily adjacent or related, and then take the 4 entries determined by those rows and columns.

Each 2×2 submatrix of a permutation matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example. $\Pi\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = cdh$

Example. $\mathrm{inv}\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) = 2$ and $\mathrm{inv}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$ and $\mathrm{inv}\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) = 3$.

Definition. The *determinant* of an $n \times n$ matrix A is the number given by the formula

$$\boxed{\det A = \sum_{X \in S_n} \Pi(X, A)(-1)^{\mathrm{inv}(X)}}$$

This general formula simplifies to the following expressions for $n = 1, 2, 3$:

$\det \begin{bmatrix} a \end{bmatrix} = a.$

$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - ef).$

For $n \geq 4$, our formula $\det A$ is a sum with at least 24 terms, and so is not easy to compute by hand. We will describe a better way of computing determinants today.

The most important properties of the determinant are described by the following theorem:

Theorem. The determinant is the unique function $\det : \{n \times n \text{ matrices}\} \to \mathbb{R}$ with these 3 properties:

(1) $\boxed{\det I_n = 1}$.

(2) If B is formed by switching two columns in an $n \times n$ matrix A, then $\boxed{\det A = -\det B}$.

(3) Suppose A, B, and C are $n \times n$ matrices with columns

$$A = \left[\begin{array}{cccc} a_1 & a_2 & \dots & a_n \end{array}\right] \quad \text{and} \quad B = \left[\begin{array}{cccc} b_1 & b_2 & \dots & b_n \end{array}\right] \quad \text{and} \quad C = \left[\begin{array}{cccc} c_1 & c_2 & \dots & c_n \end{array}\right].$$

If there is a single column i where $a_i = xb_i + yc_i$ for $x, y \in \mathbb{R}$ and in all other columns j we have $a_j = b_j = c_j$ then $\boxed{\det A = x \det B + y \det C}$.

Corollary. If A is a square matrix which is not invertible then $\det A = 0$.

Corollary. If A is a permutation matrix then $\det A = (-1)^{\text{inv}(A)}$.

Proof. Note that $\Pi(X, Y) = 0$ if X and Y are different $n \times n$ permutation matrices, but $\Pi(X, X) = 1$. $\square$

2 More properties of the determinant

Recall that A^T denotes the transpose of a matrix A (the matrix whose rows are the columns of A).

Lemma. If $X \in S_n$ then $X^T \in S_n$ and $\text{inv}(X) = \text{inv}(X^T)$.

Proof. Transposing a permutation matrix does not effect the # of 2×2 submatrices equal to $\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$. $\square$

Corollary. If A is any square matrix then $\det A = \det(A^T)$.

Proof. If $X \in S_n$ then $\Pi(X, A) = \Pi(X^T, A^T)$, so our formula for the determinant gives

$$\det A = \sum_{X \in S_n} \Pi(X, A)(-1)^{\text{inv}(X)} = \sum_{X \in S_n} \Pi(X^T, A^T)(-1)^{\text{inv}(X^T)}.$$

As X ranges over all elements of S_n, the transpose X^T also ranges over all elements of S_n, so the last sum is equal to $\sum_{X \in S_n} \Pi(X, A^T)(-1)^{\text{inv}(X)} = \det(A^T)$. $\square$

The following lemma is a weaker form of a statement we will prove later in the lecture.

Lemma. Let A and B be $n \times n$ matrices with $\det A \neq 0$. Then $\det(AB) = (\det A)(\det B)$.

Proof. Define $f : \{ n \times n \text{ matrices} \} \to \mathbb{R}$ as the function $f(M) = \frac{\det(AM)}{\det A}$.

Then f has the defining properties of the determinant, so must be equal to det since det is the unique function with these properties. In more detail:

- We have $f(I_n) = \frac{\det(AI_n)}{\det A} = \frac{\det A}{\det A} = 1$.
- If M' is given by swapping two columns in M, then AM' is given by swapping the two corresponding columns in AM, so $f(M') = \frac{\det(AM')}{\det A} = \frac{-\det(AM)}{\det A} = -f(M)$.

- If column i of M is x times column i of M' plus y times column i of M'' and all other columns of M, M', and M'' are equal, then the same is true of AM, AM', and AM'' so

$$f(M) = \frac{\det(AM)}{\det A} = \frac{x\det(AM') + y\det(AM'')}{\det A} = xf(M') + yf(M'').$$

These properties uniquely characterise det, so f and det must be the same function.

Therefore $f(B) = \frac{\det(AB)}{\det A} = \det B$ for any $n \times n$ matrix B, so $\det(AB) = (\det A)(\det B)$. $\square$

3 Determinants of triangular and invertible matrices

An $n \times n$ matrix A is *upper-triangular* if all of its nonzero entries occur in positions on or above the diagonal positions $(1, 1), (2, 2), (3, 3), \dots, (n, n)$. Such a matrix looks like

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

where the $*$ entries can be any numbers (even 0).

An $n \times n$ matrix A is *lower-triangular* if all of its nonzero entries occur in positions on or below the diagonal positions. Such a matrix looks like

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

where the $*$ entries can again be any numbers.

The transpose of an upper-triangular matrix is lower-triangular, and vice versa.

We say that a matrix is *triangular* if it is either upper- or lower-triangular.

A matrix is *diagonal* if it is both upper- and lower-triangular, i.e., has nonzero entries only on the diagonal:

$$\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

The *diagonal entries* of A are the numbers that occur in positions $(1, 1), (2, 2), (3, 3), \dots, (n, n)$.

Proposition. If A is a triangular matrix then $\det A$ is the product of the diagonal entries of A.

For example, we have $\det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc$.

Proof. Assume A is upper-triangular. If $X \in S_n$ and $X \neq I_n$ then at least one nonzero entry of X is in a position below the diagonal, in which case $\Pi(X, A)$ is a product of numbers which includes 0 (since all positions below the diagonal in A contain zeros) and is therefore 0.

Hence $\det A = \sum_{X \in S_n} \Pi(X, A)(-1)^{\mathrm{inv}(X)} = \Pi(I_n, A) =$ the product of the diagonal entries of A.

If A is lower-triangular then the same result follows since $\det A = \det(A^T)$. $\square$

Lemma. If A is an $n \times n$ matrix then $\det A$ is a nonzero multiple of $\det(\texttt{RREF}(A))$.

Proof. Suppose B is obtained from A by an elementary row operation. To prove the lemma, it is enough to show that $\det B$ is a nonzero multiple of $\det A$. There are three possibilities for B:

1. If B is formed by swapping two rows of A then $B = XA$ for a permutation matrix $X \in S_n$, so $\det B = \det(XA) = (\det X)(\det A) = \pm \det A$.
2. If B is formed by rescaling a row of A by a nonzero scalar $\lambda \in \mathbb{R}$ then $B = DA$ where D is a diagonal matrix of the form

$$D = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

   and in this case $\det D = \lambda \neq 0$, so $\det B = \det(DA) = (\det D)(\det A) = \lambda \det A$.
3. If B is formed by adding a multiple of row i of A to row j, then $B = TA$ for a triangular matrix T whose diagonal entries are all 1 and whose only other nonzero entry appears in column i and row j, so we have $\det B = \det(TA) = (\det T)(\det A) = \det A$.

This shows that performing an elementary row operation to A multiplies $\det A$ by a nonzero number. Since we obtain $\texttt{RREF}(A)$ by performing a sequence of row operations to A, it follows that $\det(\texttt{RREF}(A))$ is a sequence of nonzero numbers times $\det A$. $\square$

This brings us to a famous property of the determinant.

Theorem. An $n \times n$ matrix A is an invertible if and only if $\det A \neq 0$.

Proof. We have already seen that if A is not invertible then $\det A = 0$. If A is invertible then $\texttt{RREF}(A) = I_n$ so $\det A \neq 0$ since $\det A$ is a nonzero multiple of $\det(\texttt{RREF}(A)) = \det I_n = 1$. $\square$

Calculating the determinant is not a particularly efficient way of checking if a matrix is invertible if $n > 2$. The quickest way to compute $\det A$ involves just as much work as it takes to row reduce A to echelon form, which would also tell us if A is invertible or not.

Now that we know that $\det A \neq 0$ for all invertible matrices, we can show that the determinant is a *multiplicative function*.

Lemma. Let A and B be $n \times n$ matrices. If A or B is not invertible then AB is not invertible.

Proof. Note that $\operatorname{Col} AB \subset \operatorname{Col} A$ since if $x \in \operatorname{Col} AB$ then $x = (AB)v = A(Bv)$ for some $v \in \mathbb{R}^n$.

Also note that $\operatorname{Nul} B \subset \operatorname{Nul} AB$ since if $Bv = 0$ then $(AB)v = A(Bv) = 0$.

If A is not invertible then $\operatorname{Col} A$ is contained in but not equal to $\mathbb{R}^n$, so $\operatorname{Col} AB \neq \mathbb{R}^n$.

If B is not invertible then $\operatorname{Nul} B$ contains but is not equal to $\{0\}$, so $\operatorname{Nul} AB \neq \{0\}$.

In either case it follows that either $\operatorname{Col} AB \neq \mathbb{R}^n$ or $\operatorname{Nul} AB \neq \{0\}$ so AB is not invertible. $\square$

Theorem. If A and B are any $n \times n$ matrices then $\det(AB) = (\det A)(\det B)$.

Proof. We already proved this in the case when $\det A \neq 0$.

If $\det A = 0$, then A is not invertible, so by the previous lemma AB is not invertible either, so

$$\det(AB) = 0 = (\det A)(\det B).$$

□

It is very difficult to derive this theorem directly from the formula $\det A = \sum_{X \in S_n} \Pi(X, A)(-1)^{\mathrm{inv}(X)}$. So as not to doubt this surprising property, let's try to verify it in an example.

Example. We have $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2$ and $\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 10 - 12 = -2$.

On the other hand, $\det \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \right) = \det \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix} = 290 - 286 = 4.$

4 Computing determinants

Our proof that $\det A$ is a nonzero multiple of $\det(\texttt{RREF}(A))$ can be turned into an effective algorithm for computing the determinant.

Algorithm to compute $\det A$.

Input: an $n \times n$ matrix A.

1. Start by setting $D = 1$.
2. Row reduce A to an echelon form E. (It is not necessary to bring A all the way to *reduced* echelon form: we just need to row reduce A until we get an upper triangular matrix.) Each time you perform a row operation in this process, modify the number D as follows:
   (a) When you switch two rows, multiply D by -1.
   (b) When you rescale a row by a nonzero factor λ, multiply D by λ.
   (c) When you add a multiple of a row to another row, don't do anything to D.

The determinant $\det E$ is the product of the diagonal entries of E

The determinant of A is given by $\det A = (\det E)/D$.

The easiest way to understand this algorithm is through an example.

Example. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 0 & -4 \\ 2 & 4 & 6 \end{bmatrix}$.

To compute its determinant, we row reduce to echelon form which keeping track of the factor D:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 0 & -4 \\ 2 & 4 & 6 \end{bmatrix} \qquad D = 1$$

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -9 \\ 2 & 4 & 6 \end{bmatrix} \qquad \text{(we added a multiple of row one to row two)} \quad D = 1$$

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -9 \\ 0 & -2 & -4 \end{bmatrix} \qquad \text{(we added a multiple of row one to row three)} \quad D = 1$$

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & -2 & -4 \end{bmatrix} \qquad \text{(we multiplied row two by } -1/3) \quad D = -1/3$$

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} = E \quad \text{(we added a multiple of row two to row three)} \quad D = -1/3$$

We then get $\det A = (\det E)/D = (1 \cdot 1 \cdot 2)/(-1/3) = -6$.

This agrees with our earlier for the determinant of a 3-by-3 matrix, which gives

$$\det A = 1(0 - (-16)) - 3(6 - (-8)) + 5(4 - 0) = 16 - 3(14) + 5(4) = 16 - 42 + 20 = -6.$$

Another sometimes useful algorithm to compute $\det A$.

Given an $n \times n$ matrix A, define $A^{(i,j)}$ as the $(n-1) \times (n-1)$ submatrix formed by removing row i and column j from A.

Example. If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A^{(1,2)} = \begin{bmatrix} d & f \\ g & i \end{bmatrix}$.

Theorem. If A is the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

then

$$\boxed{\det A = a_{11} \det A^{(1,1)} - a_{12} \det A^{(1,2)} + a_{13} \det A^{(1,3)} - \dots - (-1)^n a_{1n} \det A^{(1,n)}}$$

and also

$$\boxed{\det A = a_{11} \det A^{(1,1)} - a_{21} \det A^{(2,1)} + a_{31} \det A^{(3,1)} - \dots - (-1)^n a_{n1} \det A^{(n,1)}.}$$

Note that each $A^{(1,j)}$ or $A^{(j,1)}$ is a square matrix smaller than A, so $\det A^{(1,j)}$ or $\det A^{(j,1)}$ can be computed by the same formula on a smaller scale.

Proof. The second formula follows from the first formula since $\det A = \det(A^T)$. (Why?)

The first formula is a consequence of the formula for $\det A$ we derived last lecture. One needs to show

$$-(-1)^j a_{1j} \det A^{(1,j)} = \sum_{X \in S_n^{(j)}} \Pi(X, A)(-1)^{\mathrm{inv}(X)}$$

where $S_n^{(j)}$ is the set of $n \times n$ permutation matrices which have a 1 in column j of the first row. Summing the left expression over $j = 1, 2, \ldots, n$ gives the desired formula, while summing the right expression over $j = 1, 2, \ldots, n$ gives $\sum_{X \in S_n} \Pi(X, A)(-1)^{\mathrm{inv}(X)} = \det A$. $\square$

Example. This result can be used to derive our formula for the determinant of a 3-by-3 matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} = a(ef - hi) - b(di - fg) + c(dh - eg).$$

For anything larger than a 3-by-3 matrix, it is usually faster to compute the determinant using row reduction.

5 Vocabulary

Keywords from today's lecture:

1. **Upper-triangular matrix**.

   A square matrix of the form $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ with zeros in all positions below the main diagonal.

2. **Lower-triangular matrix**.

   A square matrix of the form $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$ with zeros in all positions above the main diagonal.

   The transpose of an upper-triangular matrix.

3. **Triangular matrix**.

   A matrix that is either upper-triangular or lower-triangular.

4. **Diagonal matrix**.

   A square matrix of the form $\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$ with zeros in all non-diagonal positions.

   A matrix that is both upper-triangular and lower-triangular.