

1 Last time: eigenvector and eigenvalues

Everywhere in this lecture, n is a positive integer and A is an $n \times n$ matrix.

Let I denote the $n \times n$ identity matrix.

Definition. A vector $v \in \mathbb{R}^n$ is an *eigenvector* for A with *eigenvalue* $\lambda \in \mathbb{R}$ if $v \neq 0$ and $Av = \lambda v$.

The set of all $v \in \mathbb{R}^n$ with $Av = \lambda v$ is the λ -*eigenspace* of A for λ . This is just the nullspace of $A - \lambda I$.

Proposition. Let λ be a number. The following are equivalent:

1. There exists an eigenvector $v \in \mathbb{R}^n$ for A with eigenvalue λ .
2. The matrix $A - \lambda I$ is not invertible.
3. $\det(A - \lambda I) = 0$.
4. The λ -eigenspace for A contains a nonzero vector.

Let x be a variable. Then $\det(A - xI)$ is a polynomial in x , called the *characteristic polynomial* of A . The eigenvalues of A are precisely the solutions to the equation

$$\det(A - xI) = 0$$

which we call the *characteristic equation* for A .

Last time we proved two nontrivial theorems:

Theorem. The eigenvalues of a triangular square matrix A are its diagonal entries. If these numbers are d_1, d_2, \dots, d_n then the characteristic polynomial of A is $(d_1 - x)(d_2 - x) \cdots (d_n - x)$.

Theorem. Suppose $v_1, v_2, \dots, v_r \in \mathbb{R}^n$ are nonzero vectors. Assume each v_i is an eigenvector for an $n \times n$ matrix A . Let λ_i be the eigenvalue corresponding to v_i , so that $Av_i = \lambda_i v_i$. If $\lambda_1, \lambda_2, \dots, \lambda_r$ are all distinct, meaning that $\lambda_i \neq \lambda_j$ if $i \neq j$, then the vectors v_1, v_2, \dots, v_r are linearly independent.

To illustrate these results and motivate the new topics today, let's look at a detailed example.

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since A is triangular, its characteristic polynomial is $(1 - x)(2 - x)(3 - x)$ and its eigenvalues are 1, 2, 3.

1-eigenspace. The eigenvectors of A with eigenvalue 1 are the nonzero elements of $\text{Nul}(A - I)$.

$$A - I = \begin{bmatrix} 0 & 5 & 4 \\ & 1 & 0 \\ & & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ & 5 & 4 \\ & & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 4 \\ & & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix} = \text{RREF}(A - I).$$

This shows that $x \in \text{Nul}(A - I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis

for $\text{Nul}(A - I)$. Therefore all eigenvectors of A with eigenvalue 1 are nonzero scalar multiples of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

2-eigenspace. The eigenvectors of A with eigenvalue 2 are the nonzero elements of $\text{Nul}(A - 2I)$.

$$A - 2I = \begin{bmatrix} -1 & 5 & 4 \\ & 0 & 0 \\ & & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix} = \text{RREF}(A - 2I).$$

This shows that $x \in \text{Nul}(A - 2I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\text{Nul}(A - 2I)$. All eigenvectors of A with eigenvalue 2 are nonzero scalar multiples of $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

3-eigenspace. The eigenvectors of A with eigenvalue 3 are the nonzero elements of $\text{Nul}(A - 3I)$.

$$A - 3I = \begin{bmatrix} -2 & 5 & 4 \\ & -1 & 0 \\ & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 4 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ & 1 & 0 \\ & & 0 \end{bmatrix} = \text{RREF}(A - 3I).$$

This shows that $x \in \text{Nul}(A - 3I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\text{Nul}(A - 3I)$. All eigenvectors of A with eigenvalue 3 are nonzero scalar multiples of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Since 1, 2, 3, are distinct, the second theorem implies that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.

Consider the matrix whose columns are given by these linearly independent vectors:

$$P = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the columns of P are linearly independent, P is invertible. Recall that

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The product Pe_i is the i th column of P , so

$$Pe_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Pe_2 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Pe_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Since $Px = y$ means that $P^{-1}y = P^{-1}Px = Ix = x$, it follows that

$$P^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1 \quad \text{and} \quad P^{-1} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e_2 \quad \text{and} \quad P^{-1} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = e_3.$$

Combining these identities shows that

$$P^{-1}APe_1 = P^{-1}A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1.$$

$$P^{-1}APe_2 = P^{-1}A \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = 2P^{-1} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = 2e_2.$$

$$P^{-1}APe_3 = P^{-1}A \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 3P^{-1} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 3e_3.$$

These calculations determine the columns of the matrix $P^{-1}AP$.

If fact, we see that $P^{-1}AP = D$ where D is the diagonal matrix

$$D = [e_1 \quad 2e_2 \quad 3e_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

This means that $A = P(P^{-1}AP)P^{-1} = PDP^{-1}$, i.e.,

$$\begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}.$$

One application of this decomposition: we can derive a simple formula for an arbitrary power A^n of A .

Define $A^0 = I$, $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, and so on.

Lemma. For any integer $n \geq 0$ we have $A^n = (PDP^{-1})^n = PD^nP^{-1}$.

Proof. Do some small examples and convince yourself that the pattern continues:

$$A^2 = AA = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

$$A^3 = A^2A = PD^2P^{-1}PDP^{-1} = PD^2IDP^{-1} = PD^3P^{-1}$$

$$A^4 = A^3A = PD^3P^{-1}PDP^{-1} = PD^3IDP^{-1} = PD^4P^{-1}$$

⋮

and so on. □

Lemma. For any integer $n \geq 0$ we have

$$D^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}.$$

Proof. This is true since to multiply diagonal matrices we just multiply the entries in the corresponding diagonal positions:

$$\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_k \end{bmatrix} \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_k \end{bmatrix} = \begin{bmatrix} x_1y_1 & & & \\ & x_2y_2 & & \\ & & \ddots & \\ & & & x_ky_k \end{bmatrix}.$$

Therefore to evaluate $D^n = DD \cdots D$, we just raise each diagonal entry to the n th power. □

Finally, by the usual algorithm we can compute $P^{-1} = \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$.

(I'm skipping the details — check that this is the correct inverse!)

Putting everything together gives the identity

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5 \cdot 2^n & 2 \cdot 3^n \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5(2^n - 1) & 2(3^n - 1) \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}. \end{aligned}$$

We've done all these calculations for their own sake as a means of illustrating some key concepts. But these calculations would also come up in the solution of the following discrete dynamical system. Suppose $a_0, a_1, a_2, \dots, b_0, b_1, b_2, \dots,$ and c_0, c_1, c_2, \dots are sequences of numbers. For each integer $n \geq 1$, suppose

$$a_n = a_{n-1} + 5b_{n-1} + 4c_{n-1} \quad \text{and} \quad b_n = 2b_{n-1} \quad \text{and} \quad c_n = 3c_{n-1}. \quad (*)$$

How could we find a formula for $a_n, b_n,$ and c_n in terms of n and the sequences' initial values a_0, b_0, c_0 ? Note that (*) is equivalent to

$$\begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} = A \begin{bmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{bmatrix} = A^2 \begin{bmatrix} a_{n-2} \\ b_{n-2} \\ c_{n-2} \end{bmatrix} = \dots = A^n \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}.$$

Thus, our formula for A^n gives

$$a_n = a_0 + 5(2^n - 1)b_0 + 2(3^n - 1)c_0 \quad \text{and} \quad b_n = 2^n b_0 \quad \text{and} \quad c_n = 3^n c_0.$$

If $a_0 = b_0 = c_0 = 1$ then $a_{10} = 123212$ and $b_{10} = 1024$ and $c_{10} = 59049$. Moreover,

$$\lim_{n \rightarrow \infty} \frac{a_n}{3^n} = \lim_{n \rightarrow \infty} \frac{a_0 + 5(2^n - 1)b_0 + 2(3^n - 1)c_0}{3^n} = 2c_0.$$

2 Similar matrices

Definition. Two $n \times n$ matrices X and Y are *similar* if there exists an invertible $n \times n$ matrix P with $X = PYP^{-1}$. In this case observe that $Y = P^{-1}PYP^{-1}P = P^{-1}XP$. If X and Y are similar, then we say that “ X is *similar to* Y ” and “ Y is *similar to* X .” (Each statement implies the other.)

In the previous example we showed that $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are similar matrices.

Proposition. An $n \times n$ matrix A is always similar to itself.

Proof. Since $I = I^{-1}$ we have $A = PAP^{-1}$ for $P = I$. □

Proposition. Suppose A, B, C are $n \times n$ matrices. Assume A and B are similar. Assume B and C are also similar. Then A and C are similar.

Proof. If $A = PBP^{-1}$ and $B = QCQ^{-1}$ where P, Q are invertible $n \times n$ matrices, then

$$A = PQCQ^{-1}P^{-1} = (PQ)C(PQ)^{-1} = RCR^{-1}$$

for the invertible matrix $R = PQ$. □

Theorem. If A and B are similar $n \times n$ matrices then A and B have the same characteristic polynomial and so have the same eigenvalues.

Proof. We just need to remember that $\det(XY) = \det(X)\det(Y)$ and $\det(I) = 1$.

If $A = PBP^{-1}$ then $A - xI = PBP^{-1} - xI = PBP^{-1} - xPIP^{-1} = P(B - xI)P^{-1}$.

Therefore if $A = PBP^{-1}$ then

$$\det(A - xI) = \det(P(B - xI)P^{-1}) = \det(P)\det(B - xI)\det(P^{-1}).$$

But note that $\det(P)\det(P^{-1}) = \det(PP^{-1}) = \det(I) = 1$, so $\det(A - xI) = \det(B - xI)$. □

Caution. Matrices may have the same eigenvalues but not be similar.

The implication goes in one direction only:

$$\text{similar} \quad \Rightarrow \quad \text{same eigenvalues.}$$

For example, the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

both have eigenvalues 2, 2 but are not similar.

Since $A = 2I$ we have $PAP^{-1} = 2PIP^{-1} = 2PP^{-1} = 2I = A \neq B$ for all invertible matrices P .

Caution. Row equivalence of matrices \neq similarity of matrices.

Row operations usually change eigenvalues, whereas similar matrices always have the same eigenvalues.

Definition. A square matrix X is *diagonalizable* if X is similar to a diagonal matrix, i.e., there exists a diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and an invertible matrix P such that $X = PDP^{-1}$.

In our long example in the last section, we saw that $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

Theorem. An $n \times n$ matrix A is diagonalizable if and only if the set of eigenvectors of A spans all of \mathbb{R}^n , or equivalently contains a subset of n linearly independent vectors.

More precisely, suppose D is an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$ and P is an $n \times n$ invertible matrix with columns v_1, v_2, \dots, v_n . Then $A = PDP^{-1}$ if and only if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A and v_1, v_2, \dots, v_n are eigenvectors of A such that $Av_i = \lambda_i v_i$ for $i = 1, 2, \dots, n$.

Proof. We have

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad P = [v_1 \ v_2 \ \cdots \ v_n].$$

Then $Pe_i = v_i$ so $P^{-1}v_i = e_i$ and $De_i = \lambda_i e_i$, so

$$PDP^{-1}v_i = PD e_i = \lambda_i P e_i = \lambda_i v_i$$

for each $i = 1, 2, \dots, n$.

Therefore if $A = PDP^{-1}$ then v_1, v_2, \dots, v_n are eigenvectors for A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. In this case, as P is invertible, the columns v_1, v_2, \dots, v_n must be linearly independent, so A has n linearly independent eigenvectors.

Conversely, suppose A has n linearly independent eigenvectors v_1, v_2, \dots, v_n with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Define

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad P = [v_1 \ v_2 \ \cdots \ v_n]$$

as before. Since $Pe_i = v_i$ and $P^{-1}v_i = e_i$, we have

$$P^{-1}APe_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i.$$

This calculates the i th column of $P^{-1}AP$. Since $\lambda_i e_i$ is also the i th column of the diagonal matrix D , we deduce that $P^{-1}AP = D$. Therefore $A = P(P^{-1}AP)P^{-1} = PDP^{-1}$ is diagonalizable. \square

Not every matrix is diagonalizable. It takes some work to decide if a given matrix is diagonalizable. Here is one easy criterion, which is sufficient but not necessary:

Corollary. An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof. Suppose A has n distinct eigenvalues. By the theorem last time, any choice of eigenvectors for A corresponding to these eigenvalues will be linearly independent, so A will have n linearly independent eigenvectors. \square

Example. The matrix $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$ is triangular so has eigenvalues 5, 0, -2.

These are distinct numbers, so A is diagonalizable.

Next time: how to “diagonalize” (that is, find P such that $A = PDP^{-1}$) a diagonalizable matrix.

3 Vocabulary

Keywords from today's lecture:

1. **Similar matrices.**

Two $n \times n$ matrices A and B are similar if there exists an invertible $n \times n$ matrix M with

$$A = MBM^{-1}.$$

If A and B are similar and B and C are similar, then A and C are similar.

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$.

2. **Diagonalizable matrix.**

A matrix that is similar to a diagonal matrix. An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors. One way this can happen is if the matrix has n distinct eigenvalues.