1 Final review

The final exam will be cumulative, covering material from Lectures 1–22.

What follows is a sketch of the key concepts from each major topic group in the course. This outline is meant as a baseline for what you should know going into the final examination.

Other material covered in the lecture notes (but not mentioned below) might appear on the test. But what's below is the stuff you should review first.

Pre-midterm topics:

1. **Linear systems.**
   (a) Know definition of a *linear system* and its *solutions*, and when two linear systems are *equivalent*.
   (b) Understand that every linear system has either 0, 1, or infinitely many solutions.
   (c) Know how to extract the *coefficient matrix* and *augmented matrix* from a linear system. Understand the difference between these two matrices.

2. **Row reduction.**
   (a) Know the definitions of the elementary row operations: (1) swapping two rows, (2) rescaling a row by a nonzero constant, (3) adding a multiple of one row to another.
   (b) Understand how row operations, performed on a matrix A, correspond to multiplying A on the right by an elementary matrix E.
   (c) Matrices are *row equivalent* if one is obtained from the other by a sequence of row operations. Each matrix is row equivalent to a unique matrix in reduced echelon form. Linear systems whose augmented matrices are row equivalent have the same solutions.
   (d) Know definitions of *echelon form*, *reduced echelon form*.
   (e) **Important:** know how to carry out the *row reduction algorithm* to reduce a matrix, quickly and without using a calculator, to its unique reduced echelon form.
   (f) Understand the meaning of the *pivot positions* and *pivot columns* of a matrix A. Remember that these cannot be determined without first reducing A to echelon form.
   (g) If A is the coefficient matrix of a linear system in variables $x_1, x_2, \dots, x_n$ then x_i is a *basic variable* if i is a pivot column of A, and otherwise is a *free variable*.
   (h) **Important:** a linear system has zero solutions if the last column in its augment matrix is a pivot column. Assume this does not happen. Then the system has infinitely many solutions if it has at least one free variable, and has exactly one solution otherwise.

3. **Vectors and matrix operations.**
   (a) A *vector* is just a list of numbers oriented as a column: $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.
   (b) $\mathbb{R}^n$ is the set of vectors with n rows.
   (c) We can add vectors with the same number of rows and multiply vectors by scalars:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix} \quad \text{and} \quad x \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} xa \\ xb \\ xc \end{bmatrix}.$$

(d) We draw vectors in $\mathbb{R}^2$ as arrows from the origin in the xy-plane. Know how to interpret the sum of two vectors in $\mathbb{R}^2$ in terms of these pictures.

(e) Know that the *zero vector* in $\mathbb{R}^n$ is the vector whose entries are all zeros: $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$.

(f) Know definitions of *linear combinations* and *span* of a set of vectors.

(g) **Important:** know how to multiply an $m \times n$ matrix A with $v \in \mathbb{R}^n$ or an $n \times k$ matrix B.

(h) If the columns of A are $a_1, a_2, \dots, a_n \in \mathbb{R}^m$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $Ax = b \in \mathbb{R}^m$ has the same solutions as the vector equation $x_1a_1 + x_2a_2 + \cdots + x_na_n = b$.

(i) **Important:** if A is an $m \times n$ matrix then the following are equivalent:

   i. $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.

   ii. Each $b \in \mathbb{R}^m$ is a linear combination of the columns of A.

   iii. The span of the columns of A is $\mathbb{R}^m$

   iv. A has a pivot position in every row.

4. **Linear independence**.

(a) Know the definition of a *linearly independent/dependent* set of vectors.

Vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ are *linearly independent* if the only way that we can have $c_1, c_2, \dots, c_p \in \mathbb{R}$ and $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$ is if $c_1 = c_2 = \cdots = c_p = 0$.

(b) **Important:** given $v_1, v_2, \dots, v_p \in \mathbb{R}^n$, form the matrix $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$. The vectors are linearly independent if and only if A has a pivot position in every column.

(c) Understand that two vectors are linearly independent if and only if neither is a scalar multiple of the other, while n vectors are linearly independent if and only if no vector is a linear combination of the others.

(d) If $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ and $p > n$, then the vectors are linearly dependent.

5. **Linear transformations**.

(a) Understand what the function notation "$f : X \to Y$" means.

(b) Know definitions of *domain*, *codomain*, *image*, *range* of function.

(c) Know definition of a *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$.

(d) Every linear function $T : \mathbb{R}^n \to \mathbb{R}^m$ has the formula $f(v) = Av$ for some $m \times n$ matrix A, called the *standard matrix* of T.

(e) **Important:** know how to compute A from T via $A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}$.

(f) Understand definitions of *one-to-one*, *onto*, and *invertible* functions.

(g) T is linear and one-to-one $\Leftrightarrow$ columns of A are linearly independent.

(h) T is linear and onto $\Leftrightarrow$ columns of A span $\mathbb{R}^m$.

(i) Understand how to add linear transformations/matrices and multiply these by scalars.

(j) Know definition of the *composition* $f \circ g$ of two functions and how this corresponds to the product AB of two matrices. What must be the sizes of matrices A and B for AB to be defined?

(k) Recall definitions of the identity matrix and matrix transpose; $(AB)^T = B^T A^T$.

6. **Inverses**.

(a) Know the definitions of an invertible matrix, and what are the definitions of the inverses of an invertible function or invertible matrix.

(b) The inverse of a linear, invertible function $\mathbb{R}^n \to \mathbb{R}^n$ is also linear.

(c) **Important:** if A is an $n \times n$ matrix then the following are equivalent:

i. A is invertible.

ii. There is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.

iii. The columns of A span $\mathbb{R}^n$.

iv. The columns of A are linearly independent.

v. $\texttt{RREF}(A) = I_n$.

vi. $\det A \neq 0$.

(d) **Important:** know how to compute A^{-1} by row reducing $\begin{bmatrix} A & I_n \end{bmatrix}$ to $\begin{bmatrix} I_n & A^{-1} \end{bmatrix}$.

(e) Know formula for inverse of 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

7. **Subspaces.**

(a) Know definitions of *subspace*, *basis*, and *dimension*.

(b) Know definitions of *column space*, *null space*, and *rank* of a matrix.

(c) **Important:** know how to compute a basis for $\operatorname{Nul} A$ and $\operatorname{Col} A$, and how to compute the dimension of these subspaces.

(d) Understand that if A has n columns then $n = \operatorname{rank} A + \dim \operatorname{Nul} A$.

(e) Basis theorem: if H is a subspace of $\mathbb{R}^n$ with dimension p, then any set of p vectors spanning H is a basis for H, and any set of p linearly independent vectors in H is a basis for H.

Post-midterm topics:

8. **Determinants.**

(a) Know various definitions of the determinant of a square matrix A:

i. $\det A = \sum_{X \in S_n} (-1)^{\operatorname{inv}(X)} \Pi(X, A)$.

ii. $\det A = A_{11} \det A^{(1,1)} - A_{12} \det A^{(1,2)} + A_{13} \det A^{(1,3)} - \dots$

iii. det is the unique function with $\det I = 1$ and two other properties. (What are they?)

(b) **Important:** know how to compute $\det A$ by row reducing, keeping track of some extra data.

(c) Remember formulas for $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

(d) Remember that the determinant of a triangular matrix is the product of its diagonal entries.

(e) Product formula: $\det(AB) = (\det A)(\det B)$ so $\det(A^{-1}) = \frac{1}{\det A}$.

(f) Understand the following statements:

Adding one column to another does not change the determinant.

Switching two columns multiplies the determinant by -1.

A matrix is invertible if and only if its determinant is nonzero.

(g) Understand how $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the area of a certain parallelogram.

9. **Vector spaces**.

(a) Review definition and examples of vector spaces in general, besides the motivating case of $\mathbb{R}^n$.

(b) Understand how notations of subspace, linear transformation, linear combination, span, linear independence, basis, and dimension extend to abstract vector spaces.

(c) Example problem: given a finite list of polynomials in one variable, find the dimension of the vector space they span.

10. **Eigenvalues and eigenvectors**.

(a) Be familiar with this definition: a vector $v \in \mathbb{R}^n$ is an eigenvector of a matrix A with eigenvalue $\lambda \in \mathbb{R}$ if $Av = \lambda v$ and $v \neq 0$.

(b) Know how to make sense of this generalization: a vector $v \in \mathbb{C}^n$ is a (complex) eigenvector of a matrix A with (complex) eigenvalue λ if $Av = \lambda v$ and $v \neq 0$.

(c) The eigenvalues of a matrix A are the roots of the characteristics polynomial $\det(A - xI)$. These may be complex numbers, and they may be repeated. The sum of these roots (with multiplicity) is the trace $\mathrm{tr}A$. Their product is $\det A$.

(d) The eigenvalues of a triangular matrix are its diagonal entries.

(e) **Important:** know how to find the eigenvalues of A by factoring $\det(A - xI)$. Given an eigenvalue λ, know how to find a corresponding eigenvector. This can be done by computing a basis for $\mathrm{Nul}(A - \lambda I)$. When this subspace is nonzero, it is called the λ-eigenspace of A.

(f) Know what the relationship is between the eigenvectors and eigenvalues of A, A^T, and A^{-1}:

i. A and A^T have same eigenvalues, usually different eigenvectors.

ii. A and A^{-1} have same eigenvectors, but reciprocal eigenvalues.

11. **Diagonalization**.

(a) Square matrices A and B are similar if $A = PBP^{-1}$ for some invertible matrix P. Similar matrices have the same size and the same eigenvalues.

(b) A matrix A is diagonalizable if it is similar to a diagonal matrix. This can happen only if A is square. An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$. In this case, it holds that $A = PDP^{-1}$ where

$$P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

and D is the diagonal matrix whose entry in position (i, i) is the eigenvalue of v_i.

(c) A useful shortcut: an $n \times n$ matrix is diagonalizable whenever it has n distinct eigenvalues. Eigenvectors with distinct eigenvalues (of a fixed matrix) are always linearly independent.

(d) **Important:** a matrix with repeated eigenvalues may still be diagonalizable. Know how to check if a matrix A is diagonalizable in general: compute the distinct eigenvalues, then compute the dimensions of the corresponding eigenspaces. The matrix is diagonalizable if and only if these dimensions sum to n where A is $n \times n$.

(e) **Important:** given a matrix A which is diagonalizable, know how to find a decomposition $A = PDP^{-1}$. Know how to use this to give an exact formula for A^k for any integer k.

12. **Inner products and diagonalization**.

(a) **Important:** know definitions and properties of the inner product $u \bullet v$, the length $\|v\|$, unit vectors, orthogonal vectors, orthonormal vectors.

(b) Know definition of the orthogonal complement V^T of a subspace $V \subset \mathbb{R}^n$. We have $\dim V + \dim V^\perp = n$ and $V \cap V^\perp = \{0\}$. If A is a matrix then $(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T$.

(c) Orthogonal projections: if $y \in \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ is a subspace, then there is a unique vector $\mathrm{proj}_W(y) \in W$ such that $y - \mathrm{proj}_W(y) \in W^\perp$. This is the orthogonal projection of y onto W.

(d) Know how to compute $\mathrm{proj}_W(y)$ given an orthogonal basis $v_1, v_1, \dots, v_p$ for W:

$$\mathrm{proj}_W(y) = \frac{y \bullet v_1}{v_1 \bullet v_1} v_1 + \frac{y \bullet v_2}{v_2 \bullet v_2} v_2 + \cdots + \frac{y \bullet v_p}{v_p \bullet v_p} v_p.$$

Know that the orthogonal projection is the vector $v \in \mathbb{R}^n$ which minimizes the length $\|y - v\|$.

(e) **Important:** review the definition of the Gram-Schmidt process. Know how to use it to convert a basis for a subspace to an orthogonal basis.

13. **Least-squares problems**.

(a) Know definition of a linear-squares solution to a linear system $Ax = b$: a vector $h \in \mathbb{R}^n$ such that $\|Ah - b\| \leq \|Ax - b\|$ for all $x \in \mathbb{R}^n$.

(b) Understand why a linear system always has at least one least-squares solution.

(c) **Important:** the least-squares solutions to $Ax = b$ are the exact solutions to $A^T Ax = A^T b$. Practice solving some least-squares problems.

(d) Know when a linear system $Ax = b$ has a unique least-squares solution: when A has linearly independent columns, or equivalently when $A^T A$ is invertible.

(e) Know how to compute lines of best fit and, more generally, how to approximate a function given data using least-squares.

14. **Symmetric matrices**.

(a) A matrix A is symmetric if $A^T = A$.

(b) Symmetric matrices have all real eigenvalues and are orthogonally diagonalizable.

(c) We can write $A = UDU^T = UDU^{-1}$ where D is a diagonal matrix and U is an orthogonal matrix (which means $U^T = U^{-1}$) if and only if A is symmetric.

Important: review how to find this factorization when $A = A^T$.

(d) If A is any matrix then $A^T A$ is symmetric with nonnegative eigenvalues.

(e) The singular values of A are the square roots of the eigenvalues of $A^T A$.

(f) The number of nonzero singular values of A is equal to $\mathrm{rank}\, A$.

(g) Every $m \times n$ matrix A has a *singular value decomposition*: a factorization $A = U\Sigma V^T$ where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is the unique $m \times n$ matrix whose entries in positions $(1, 1), (2, 2), \dots, (r, r)$ are the decreasing list of nonzero singular values of A, and which has 0 in all other positions.

(h) **Important:** know how to construct a singular value decomposition for a matrix A.

(i) Know how to construct the *pseudo-inverse* A^+ of A from an SVD $A = U\Sigma V^T$: form Σ^+ by transposing Σ and replacing all nonzero entries by their reciprocals; then $A^+ = V\Sigma^+ U^T$.