FINAL REVIEW – OPTIONAL ASSIGNMENT #3 – MATH 2121, FALL 2019.

Name:

Student ID:

Email:

Lecture: L1 (Wed/Fri) L2 (Mon/Wed)

Tutorial: T1A T1B T1C T2A T2B T2C

These exercises are intended to help you review for our final examination.

This is the third of 4 optional review assignments.

If you bring your solutions to the assignments to the final, then we will grade your answers and you can earn up to 5 extra credit points on the exam.

You may use any available resources, including other people, but you should write up your answers individually.

Feel free to write your solutions on your own paper, but include all of the information requested on this title page and make sure to staple your answers together.

Exercise 1. Find all (possibly complex) eigenvalues for the matrices

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Are these matrices similar?

Solution:

Exercise 2. Diagonalize the matrix

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}.$$

In other words, find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

Use this to compute exact formulas for the functions defined by

$$\begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} = A^n$$

for positive integers $n = 1, 2, 3, \dots$.

Finally, calculate the limit $\lim_{n\to\infty} A^n$.

Solution:

Exercise 3. Find the rank and eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution:

Exercise 4. Find the eigenvalues and determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Solution:

Exercise 5. Consider the vector space

$$V = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

of polynomials in one variable x with degree at most three.

(a) Define $T : V \to V$ to be the function with $T(f(x)) = f(x+1)$ for $f \in V$, so

$$T(3x) = 3x + 3 \quad \text{and} \quad T(x^2) = x^2 + 2x + 1,$$

for example. Explain why this function is linear.

(b) Let $A : \mathbb{R}^3 \to V$ and $B : V \to \mathbb{R}^3$ be the linear functions with

$$A(e_i) = x^{i-1} \quad \text{and} \quad B(x^{i-1}) = e_i \quad \text{for } i \in \{1, 2, 3\}$$

where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The composition $F = B \circ T \circ A$ is a linear function $\mathbb{R}^3 \to \mathbb{R}^3$.
Determine the standard matrix of F.

(c) Using part (b), find all eigenvalues for T and for each eigenvalue find a corresponding eigenvector.

In this context, an eigenvector for T with eigenvalue λ is a nonzero polynomial $f(x) = ax^2 + bx + c \in V$ such that

$$T(f(x)) = f(x+1) = \lambda f(x)$$

which is equivalent to

$$a(x+1)^2 + b(x+1) + c = (\lambda a)x^2 + (\lambda b)x + (\lambda c).$$

Solution: